In *Nonlinear Elasticity: Theory and Applications*
(edited by Y.B. Fu and R.W. Ogden)
Cambridge University Press (2001), pp. 1-58

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Elements of the theory of finite elasticity

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In this chapter we provide a brief overview of the main ingredients of the nonlinear theory of elasticity in order to establish the basic background material as a reference source for the other, more specialized, chapters in this volume.

1.1 Introduction

In this introductory chapter we summarize the basic equations of nonlinear elasticity theory as a point of departure and as a reference source for the other articles in this volume which are concerned with more specific topics.

There are several texts and monographs which deal with the subject of nonlinear elasticity in some detail and from different standpoints. The most important of these are, in chronological order of the publication of the first edition, Green and Zerna (1954, 1968, 1992), Green and Adkins (1960, 1970), Truesdell and Noll (1965), Wang and Truesdell (1973), Chadwick (1976, 1999), Marsden and Hughes (1983, 1994), Ogden (1984a, 1997), Ciarlet (1988) and Antman (1995). See also the textbook by Holzapfel (2000), which deals with viscoelasticity and other aspects of nonlinear solid mechanics as well as containing an extensive treatment of nonlinear elasticity. These books may be referred to for more detailed study. Subsequently in this chapter we shall refer to the most recent editions of these works. The review articles by Spencer (1970) and Beatty (1987) are also valuable sources of reference.

Section 1.2 of this chapter is concerned with laying down the basic equations of elastostatics and it includes a summary of the relevant geometry of deformation and strain, an account of stress and stress tensors, the equilibrium equations and boundary conditions and an introduction to the formulation of constitutive laws for elastic materials, with discussion of the important notions of objectivity and material symmetry. Some emphasis is placed on the special case of isotropic elastic materials, and the constitutive laws for anisotropic material consisting of one or two families of fibres are also discussed. The modifications to the constitutive laws when internal constraints such as incompressibility and inextensibility are present are provided. The general boundary-value problem of nonlinear elasticity is then formulated and the circumstances when this can be cast in a variational structure are discussed briefly.

In Section 1.3 some basic examples of boundary-value problems are given. Specifically, the equations governing some homogeneous deformations are highlighted, with the emphasis on incompressible materials. Other chapters in this volume will discuss a range of different boundary-value problems involving non-homogeneous deformations so here we focus attention on just one problem as an exemplar. This is the problem of extension and inflation of a thick-walled circular cylindrical tube. The analysis is given for an incompressible isotropic elastic solid and also for a material with two mechanically equivalent symmetrically disposed families of fibres in order to illustrate some differences between isotropic and anisotropic response.

The (linearized) equations of incremental elasticity associated with small deformations superimposed on a finite deformation are summarized in Section 1.4. The incremental constitutive law for an elastic material is used to identify the (fourth-order) tensor of elastic moduli associated with the stress and deformation variables used in the formulation of the governing equations, and explicit expressions for the components of this tensor are given in the case of an isotropic material. For the two-dimensional specialization, necessary and sufficient conditions on these components for the strong ellipticity inequalities to hold are given for both unconstrained and incompressible materials. A brief discussion of incremental uniqueness and stability is then given in the context of the dead-load boundary-value problem and the associated local inequalities are given explicit form for an isotropic material, again for both unconstrained and incompressible materials. A short discussion of global aspects of non-uniqueness for an isotropic material sets the incremental results in a broader context.

In Section 1.5 the equations of incremental deformations and equilibrium given in Section 1.4 are specialized to the plane strain context in order to provide a formulation for the analysis of incremental plane strain boundary-value problems. Specifically, we provide an example of a typical incremental boundary-value problem by considering bifurcation of a uniformly deformed half-space from a homogeneously deformed configuration into a non-homogeneous local mode of deformation. An explicit bifurcation condition is given for this problem and the results are illustrated for two forms of strain-energy function.

Finally, in Section 1.6 we summarize the equations associated with the (non-linear) dynamics of an elastic body at finite strain. The (linearized) equations for small motions superimposed on a static finite deformation are then given and these are applied to the analysis of plane waves propagating in a homogeneously deformed material.

References are given throughout the text but these are not intended to provide an exhaustive list of original sources. Where appropriate we mention papers and books where more detailed citations can be found. Also, where a topic is to be dealt with in detail in one of the other chapters of this volume the appropriate citations are included there.

1.2 Elastostatics

In this section we summarize the basic equations of the static theory of nonlinear elasticity, including the kinematics of deformation, the analysis of stress and the governing equations of equilibrium, and we introduce the various forms of constitutive law for an elastic material, including a discussion of isotropy and anisotropy. We then formulate the basic boundary-value problem of nonlinear elasticity. The development here is a synthesis of the essential material taken from the book by Ogden (1997) with some minor differences and additions.

1.2.1 Deformation and strain

We consider a continuous body which occupies a connected open subset of a three-dimensional Euclidean point space, and we refer to such a subset as a *configuration* of the body. We identify an arbitrary configuration as a *reference configuration* and denote this by $\mathcal{B}_r$. Let points in $\mathcal{B}_r$ be labelled by their position vectors $\mathbf{X}$ relative to an arbitrarily chosen origin and let $\partial\mathcal{B}_r$ denote the boundary of $\mathcal{B}_r$. Now suppose that the body is deformed quasi-statically from $\mathcal{B}_r$ so that it occupies a new configuration, $\mathcal{B}$ say, with boundary $\partial\mathcal{B}$. We refer to $\mathcal{B}$ as the *current* or *deformed configuration* of the body. The deformation is represented by the mapping $\chi : \mathcal{B}_r \to \mathcal{B}$ which takes points $\mathbf{X}$ in $\mathcal{B}_r$ to points $\mathbf{x}$ in $\mathcal{B}$. Thus,

$$\mathbf{x} = \chi(\mathbf{X}), \qquad \mathbf{X} \in \mathcal{B}_r, \tag{2.1}$$

where $\mathbf{x}$ is the position vector of the point $\mathbf{X}$ in $\mathcal{B}$. The mapping χ is called the *deformation* from $\mathcal{B}_r$ to $\mathcal{B}$. We require χ to be one-to-one and we write its inverse as χ^{-1}, so that

$$\mathbf{X} = \chi^{-1}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}. \tag{2.2}$$

Both χ and its inverse are assumed to satisfy appropriate regularity conditions. Here, it suffices to take χ to be twice continuously differentiable, but different requirements may be specified in other chapters of this volume.

For simplicity we consider only Cartesian coordinate systems and let $\mathbf{X}$ and $\mathbf{x}$ respectively have coordinates X_α and x_i, where $\alpha, i \in \{1,2,3\}$, so that $x_i = \chi_i(X_\alpha)$. Greek and Roman indices refer, respectively, to $\mathcal{B}_r$ and $\mathcal{B}$ and the usual summation convention for repeated indices is used.

The *deformation gradient tensor*, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} \tag{2.3}$$

and has Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, Grad being the gradient operator in $\mathcal{B}_r$. Local invertibility of χ requires that $\mathbf{F}$ be non-singular, and we adopt the usual convention that $\det \mathbf{F} > 0$. Similarly, for the inverse deformation gradient

$$\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{X}, \quad (\mathbf{F}^{-1})_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \tag{2.4}$$

where grad is the gradient operator in $\mathcal{B}$. With use of the notation defined by

$$J = \det \mathbf{F} \tag{2.5}$$

we then have

$$0 < J < \infty. \tag{2.6}$$

The equation

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X} \tag{2.7}$$

(in components $\mathrm{d}x_i = F_{i\alpha}\mathrm{d}X_\alpha$) describes how an infinitesimal *line element* $\mathrm{d}\mathbf{X}$ of material at the point $\mathbf{X}$ transforms *linearly* under the deformation into the line element $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$.

We now set down how elements of surface area and volume transform. Let $\mathrm{d}\mathbf{A} \equiv \mathbf{N}\mathrm{d}A$ denote a vector surface area element on $\partial\mathcal{B}_r$, where $\mathbf{N}$ is the unit outward normal to the surface, and $\mathrm{d}\mathbf{a} \equiv \mathbf{n}\mathrm{d}a$ the corresponding area element on $\partial\mathcal{B}$. Then, the area elements are connected according to *Nanson's formula*

$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A, \tag{2.8}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$ and ^{T} denotes the transpose. Note that, unlike a line element, the normal vector is not embedded in the material, i.e. $\mathbf{n}$ is not in general aligned with the same line element of material as $\mathbf{N}$.

If $\mathrm{d}V$ and $\mathrm{d}v$ denote volume elements in $\mathcal{B}_r$ and $\mathcal{B}$ respectively then we also have

$$\mathrm{d}v = J\mathrm{d}V. \tag{2.9}$$

For a volume preserving (*isochoric*) deformation we have

$$J = \det \mathbf{F} = 1. \tag{2.10}$$

A material for which (2.10) is constrained to be satisfied for all deformation gradients $\mathbf{F}$ is said to be *incompressible*.

The identities

$$\operatorname{Div}(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \operatorname{div}(J^{-1}\mathbf{F}) = \mathbf{0} \tag{2.11}$$

are important tools in transformations between equations associated with the reference and current configurations, where Div and div are the divergence operators in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. The first identity in (2.11) can readily be established by integrating (2.8) over an arbitrary closed surface in $\mathcal{B}$ and applying the divergence theorem and the second similarly by integrating $\mathbf{N}\mathrm{d}A$ over an arbitrary closed surface in $\mathcal{B}_r$.

From (2.7) we have

$$|\mathrm{d}\mathbf{x}|^2 = (\mathbf{FM}) \cdot (\mathbf{FM})\,|\mathrm{d}\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{FM}) \cdot \mathbf{M}\,|\mathrm{d}\mathbf{X}|^2, \tag{2.12}$$

where we have introduced the unit vector $\mathbf{M}$ in the direction of $\mathrm{d}\mathbf{X}$ and $\cdot$ signifies the scalar product of two vectors. Then, the ratio $|\mathrm{d}\mathbf{x}|/|\mathrm{d}\mathbf{X}|$ of the lengths of a line element in the deformed and reference configurations is given by

$$\frac{|\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{X}|} = |\mathbf{FM}| = [\mathbf{M} \cdot (\mathbf{F}^T\mathbf{FM})]^{1/2} \equiv \lambda(\mathbf{M}). \tag{2.13}$$

Equation (2.13) defines the *stretch* $\lambda(\mathbf{M})$ *in the direction* $\mathbf{M}$ at $\mathbf{X}$, and we note that it is restricted according to the inequalities

$$0 < \lambda(\mathbf{M}) < \infty. \tag{2.14}$$

If there is no stretch in the direction $\mathbf{M}$ then $\lambda(\mathbf{M}) = 1$ and hence

$$(\mathbf{F}^T\mathbf{FM}) \cdot \mathbf{M} = 1. \tag{2.15}$$

If there is no stretch in any direction, i.e. (2.15) holds for all $\mathbf{M}$, then the material is said to be *unstrained* at $\mathbf{X}$, and it follows that $\mathbf{F}^T\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor. A suitable tensor measure of strain is therefore $\mathbf{F}^T\mathbf{F} - \mathbf{I}$

since this tensor vanishes when the material is unstrained. This leads to the definition of the *Green strain tensor*

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \qquad (2.16)$$

where the 1/2 is a normalization factor. If, for a given $\mathbf{M}$, equation (2.15) holds for all possible deformation gradients $\mathbf{F}$ then the considered material is said to be *inextensible* in the direction $\mathbf{M}$.

The deformation gradient can be decomposed according to the *polar decompositions*

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \qquad (2.17)$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$, $\mathbf{V}$ are positive definite and symmetric tensors. Each of the decompositions in (2.17) is unique. Respectively, $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors.*

These stretch tensors can also be put in spectral form. For $\mathbf{U}$ we have the *spectral decomposition*

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \qquad (2.18)$$

where $\lambda_i > 0$, $i \in \{1,2,3\}$, are the *principal stretches,* $\mathbf{u}^{(i)}$, the (unit) eigenvectors of $\mathbf{U}$, are called the *Lagrangian principal axes* and $\otimes$ denotes the tensor product. Note that $\lambda(\mathbf{u}^{(i)}) = \lambda_i$ in accordance with the definition (2.13). Similarly, $\mathbf{V}$ has the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \qquad (2.19)$$

where

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}, \quad i \in \{1,2,3\}. \qquad (2.20)$$

It follows from (2.5), (2.17) and (2.18) that

$$J = \lambda_1 \lambda_2 \lambda_3. \qquad (2.21)$$

Using the polar decompositions (2.17) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \qquad (2.22)$$

These define $\mathbf{C}$ and $\mathbf{B}$, which are called, respectively, the *right* and *left Cauchy-Green deformation tensors.*

More general classes of strain tensors, i.e. tensors which vanish when there is no strain, can be constructed on the basis that $\mathbf{U} = \mathbf{I}$ when the material is unstrained. Thus, for example, we define Lagrangian strain tensors

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \quad m \neq 0, \tag{2.23}$$

$$\mathbf{E}^{(0)} = \ln \mathbf{U}, \quad m = 0, \tag{2.24}$$

where m is a real number (not necessarily an integer). Eulerian strain tensors based on the use of $\mathbf{V}$ may be constructed similarly. See, for example, Doyle and Ericksen (1956), Seth (1964) and Hill (1968, 1970, 1978). Note that for $m = 2$ equation (2.23) reduces to the Green strain tensor (2.16). For discussion of the logarithmic strain tensor (2.24) we refer to, for example, Hoger (1987).

Let ρ_r and ρ be the *mass densities* in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. Then, since the material in the volume element $\mathrm{d}V$ is the same as that in $\mathrm{d}v$ the mass is conserved, i.e. $\rho \mathrm{d}v = \rho_r \mathrm{d}V$, and hence, from (2.9), we may express the *mass conservation equation* in the form

$$\rho_r = \rho J. \tag{2.25}$$

1.2.2 Stress tensors and equilibrium equations

The surface force per unit area (or *stress vector*) on the vector area element $\mathrm{d}\mathbf{a}$ is denoted by $\mathbf{t}$. It depends on $\mathbf{n}$ according to the formula

$$\mathbf{t} = \boldsymbol{\sigma}^T \mathbf{n}, \tag{2.26}$$

where $\boldsymbol{\sigma}$, a second-order tensor independent of $\mathbf{n}$, is called the *Cauchy stress tensor*.

By means of (2.8) the force on $\mathrm{d}\mathbf{a}$ may be written as

$$\mathbf{t}\mathrm{d}a = \mathbf{S}^T \mathbf{N} \mathrm{d}A, \tag{2.27}$$

where the *nominal stress tensor* $\mathbf{S}$ is related to $\boldsymbol{\sigma}$ by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.28}$$

The *first Piola-Kirchhoff stress tensor*, denoted here by $\boldsymbol{\pi}$, is given by $\boldsymbol{\pi} = \mathbf{S}^T$ and this will be used in preference to $\mathbf{S}$ in some parts of this volume.

Let $\mathbf{b}$ denote the body force per unit mass. Then, in integral form, the *equilibrium equation* for the body may be written with reference either to $\mathcal{B}$ or $\mathcal{B}_r$. Thus,

$$\int_{\mathcal{B}} \rho \mathbf{b} \mathrm{d}v + \int_{\partial\mathcal{B}} \boldsymbol{\sigma}^T \mathbf{n} \mathrm{d}a = \int_{\mathcal{B}_r} \rho_r \mathbf{b} \mathrm{d}V + \int_{\partial\mathcal{B}_r} \mathbf{S}^T \mathbf{N} \mathrm{d}A = \mathbf{0}. \tag{2.29}$$

On use of the divergence theorem equations (2.29) yield the equivalent equilibrium equations

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \mathbf{0}, \tag{2.30}$$

$$\operatorname{Div}\mathbf{S} + \rho_r\mathbf{b} = \mathbf{0}, \tag{2.31}$$

where again div and Div denote the divergence operators in $\mathcal{B}$ and $\mathcal{B}_r$ respectively. The derivation of the pointwise equations (2.30) and (2.31) requires that the left-hand sides of these equations are continuous (in $\mathcal{B}$ and $\mathcal{B}_r$ respectively). Note that on use of (2.11) and (2.25) equation (2.31) may be converted immediately to (2.30). In components, (2.31) has the form

$$\frac{\partial S_{\alpha i}}{\partial X_\alpha} + \rho_r b_i = 0, \tag{2.32}$$

and similarly for (2.30), where $S_{\alpha i}$ are the components of $\mathbf{S}$ and b_i those of $\mathbf{b}$.

Balance of the moments of the forces acting on the body yields simply $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$, which may also be expressed as

$$\mathbf{S}^T\mathbf{F}^T = \mathbf{F}\mathbf{S}. \tag{2.33}$$

The Lagrangian formulation based on the use of $\mathbf{S}$ and equation (2.31), with $\mathbf{X}$ as the independent variable, is normally preferred in nonlinear elasticity to the Eulerian formulation based on use of $\boldsymbol{\sigma}$ and equation (2.30) with $\mathbf{x}$ as the independent variable since the initial geometry is known, whereas $\mathbf{x}$ depends on the deformation to be determined.

We now consider the work done by the surface and body forces in a virtual displacement $\delta\mathbf{x}$ from the current configuration $\mathcal{B}$. By using the divergence theorem and equation (2.31) we obtain the *virtual work* equation

$$\int_{\mathcal{B}_r} \rho_r\mathbf{b}\cdot\delta\mathbf{x}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{S}^T\mathbf{N})\cdot\delta\mathbf{x}\,\mathrm{d}A = \int_{\mathcal{B}_r} \operatorname{tr}(\mathbf{S}\delta\mathbf{F})\,\mathrm{d}V, \tag{2.34}$$

where the left-hand side of (2.34) represents the virtual work of the body and surface forces and in the integrand on the right-hand side tr denotes the trace of a second-order tensor and $\delta\mathbf{F} = \operatorname{Grad}\delta\mathbf{x}$. The term on the right-hand side is the virtual work of the stresses in the bulk of the material. For a conservative system this latter work is recoverable and is stored as elastic strain energy (this will be discussed in Section 1.2.5.1) but in general it includes a dissipative part. In either case the integrand, which represents the virtual work increment per unit volume in $\mathcal{B}_r$, may be expressed in many alternative forms using different deformation and strain measures.

For example, using (2.16), (2.17) and the symmetry (2.33), we obtain

$$\operatorname{tr}(\mathbf{S}\delta\mathbf{F}) = \operatorname{tr}(\mathbf{T}^{(1)}\delta\mathbf{U}) = \operatorname{tr}(\mathbf{T}^{(2)}\delta\mathbf{E}), \tag{2.35}$$

in which we have defined the *Biot stress tensor* $\mathbf{T}^{(1)}$ (Biot, 1965) and the *second Piola-Kirchhoff stress tensor* $\mathbf{T}^{(2)}$ (both symmetric) by

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{SR} + \mathbf{R}^T\mathbf{S}^T), \quad \mathbf{T}^{(2)} = \mathbf{SF}^{-T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{2.36}$$

We note the connection

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}). \tag{2.37}$$

More generally, the expression in (2.35) may be written in terms of the strain tensors $\mathbf{E}^{(m)}$ given by (2.23) and (2.24) and their (symmetric) *conjugate stress tensors* $\mathbf{T}^{(m)}$ as

$$\mathrm{tr}\,(\mathbf{T}^{(m)}\delta\mathbf{E}^{(m)}). \tag{2.38}$$

Note that the examples $m = 1$ and $m = 2$ from (2.35) are included in (2.38) as special cases. The notion of conjugate stress and strain tensors was introduced by Hill (1968) and applies more generally than to the special class of strain tensors (2.23). A more detailed discussion can be found in Ogden (1997). We observe that the definition of conjugate stress and strain tensors is independent of any choice of material constitutive law.

1.2.3 Elasticity

The constitutive equation of an elastic material is given in the form

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}), \tag{2.39}$$

where $\mathbf{G}$ is a *symmetric tensor-valued function*, defined on the space of deformation gradients $\mathbf{F}$. In general the form of $\mathbf{G}$ depends on the choice of reference configuration and $\mathbf{G}$ is referred to as the *response function* of the material *relative* to $\mathcal{B}_r$. For a given $\mathcal{B}_r$, therefore, the stress in $\mathcal{B}$ at a (material) point $\mathbf{X}$ depends only on the deformation gradient at $\mathbf{X}$ and not on the history of deformation. A material whose constitutive law has the form (2.39) is generally referred to as a *Cauchy elastic material*. Its specialization to the situation when there exists a strain-energy function will be considered in Section 1.2.4.

If the stress vanishes in $\mathcal{B}_r$ then

$$\mathbf{G}(\mathbf{I}) = \mathbf{O}, \tag{2.40}$$

and $\mathcal{B}_r$ is called a *natural configuration*. If the stress does not vanish in $\mathcal{B}_r$ then there is said to be *residual stress* in this configuration. In a residually-stressed configuration the traction must vanish at all points of the boundary, so that *a fortiori* residual stress is inhomogeneous in character. For detailed discussion of residual stress we refer to the work of Hoger and co-workers (see, for example, Hoger, 1985, 1986, 1993a, b and Johnson and Hoger 1993, 1995, 1998).

1.2.3.1 Objectivity

Suppose that a rigid-body deformation

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{c} \tag{2.41}$$

is superimposed on the deformation $\mathbf{x} = \chi(\mathbf{X})$, where $\mathbf{Q}$ and $\mathbf{c}$ are constants, $\mathbf{Q}$ being a rotation tensor and $\mathbf{c}$ a translation vector. Then, the resulting deformation gradient, $\mathbf{F}^*$ say, is given by

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}. \tag{2.42}$$

For an elastic material with response function $\mathbf{G}$ relative to $\mathcal{B}_r$, the Cauchy stress tensor, $\boldsymbol{\sigma}^*$ say, associated with the deformation gradient $\mathbf{F}^*$ is $\boldsymbol{\sigma}^* = \mathbf{G}(\mathbf{F}^*)$.

Under the transformation (2.41) $\boldsymbol{\sigma}$ transforms according to the formula

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T. \tag{2.43}$$

The response function $\mathbf{G}$ must therefore satisfy the *invariance requirement*

$$\mathbf{G}(\mathbf{F}^*) \equiv \mathbf{G}(\mathbf{Q}\mathbf{F}) = \mathbf{Q}\mathbf{G}(\mathbf{F})\mathbf{Q}^T \tag{2.44}$$

for each deformation gradient $\mathbf{F}$ and *all* rotations $\mathbf{Q}$. This expresses the fact that the constitutive law (2.39) is *objective*. The terminology *material frame-indifference* is also used for this concept of objectivity (see, for example, Truesdell and Noll, 1965). In essence, this means that material properties are independent of superimposed rigid-body deformations.

A second-order Eulerian tensor, such as $\boldsymbol{\sigma}$, which satisfies the transformation rule (2.43) is said to be an (Eulerian) *objective second-order tensor*. We now expand on this notion slightly. Let $\phi, \mathbf{u}, \mathbf{T}$ be (Eulerian) scalar, vector and (second-order) tensor functions defined on $\mathcal{B}$. Let $\phi^*, \mathbf{u}^*, \mathbf{T}^*$ be the corresponding functions defined on $\mathcal{B}^*$, where $\mathcal{B}^*$ is obtained from $\mathcal{B}$ by the rigid deformation (2.41). The functions are said to be (Eulerian) *objective scalar, vector and tensor functions* (or fields) if, for all such deformations,

$$\phi^* = \phi, \quad \mathbf{u}^* = \mathbf{Q}\mathbf{u}, \quad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \tag{2.45}$$

We observe that the density ρ is an example of an objective scalar function and that the normal vector $\mathbf{n}$, which appears in (2.8), and the traction vector $\mathbf{t}$, given by (2.26), are examples of objective vector functions, while the left Cauchy-green deformation tensor $\mathbf{B}$ is an objective tensor function.

It is important to distinguish between the behaviour of Lagrangian and Eulerian vector and tensor functions as far the definition of objectivity is concerned. The vector function $\mathbf{N}$, which is related to $\mathbf{n}$ by (2.8), and the right Cauchy-Green deformation tensor $\mathbf{C}$, given by (2.22), for example, are unchanged under the transformation (2.41). They are Lagrangian functions defined on $\mathcal{B}_r$. Thus, objectivity may equally well be defined in terms of Lagrangian functions. An objective Lagrangian (scalar, vector or tensor) function is one which is *unchanged* by the transformation (2.41). Other examples of objective Lagrangian tensors are the Biot and second Piola-Kirchhoff stress tensors defined in (2.36). Objective mixed tensors, such as $\mathbf{F}$, which are partly Lagrangian and partly Eulerian, change either as in (2.42) or its transpose. Thus, the nominal stress tensor $\mathbf{S}$, given by (2.28), transforms like $\mathbf{S}^* = \mathbf{S}\mathbf{Q}^T$ (for more detailed discussion, see Ogden, 1984b).

We mention here that Lagrangian vectors and tensors can be transformed into Eulerian vectors and tensors by appropriate 'push-forward' operations and this process is reversed by 'pull-back' transformations in the sense described in Marsden and Hughes (1994); see also Holzapfel (2000). The form of the push-forward and pull-back transformations depends on whether the vectors and tensors in question have covariant or contravariant character. For example, the push forward of the (covariant) Green strain tensor $\mathbf{E}$ is $\mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1}$, which is an Eulerian strain tensor, while the push forward of the (contravariant) second Piola-Kirchhoff stress tensor $\mathbf{T}^{(2)}$ is $\mathbf{F}\mathbf{T}^{(2)}\mathbf{F}^T$, which is just J times the (Eulerian) Cauchy stress tensor. Partial push forward or pull back can be applied to either type of tensor to obtain mixed tensors or to mixed tensors to obtain Lagrangian or Eulerian tensors.

1.2.3.2 Material symmetry

Let $\boldsymbol{\sigma}$ be the stress in configuration $\mathcal{B}$, and let $\mathbf{F}$ and $\mathbf{F}'$ be the deformation gradients in $\mathcal{B}$ relative to two different reference configurations, $\mathcal{B}_r$ and $\mathcal{B}'_r$ respectively. We denote by $\mathbf{G}$ and $\mathbf{G}'$ the response functions relative to $\mathcal{B}_r$ and $\mathcal{B}'_r$, so that

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) = \mathbf{G}'(\mathbf{F}'). \tag{2.46}$$

Let $\mathbf{P} = \operatorname{Grad}\mathbf{X}'$ be the deformation gradient of $\mathcal{B}'_r$ relative to $\mathcal{B}_r$, where $\mathbf{X}'$ is the position vector of a point in $\mathcal{B}'_r$. Then

$$\mathbf{F} = \mathbf{F}'\mathbf{P}. \tag{2.47}$$

Substitution of (2.47) into (2.46) then gives $\mathbf{G}(\mathbf{F}'\mathbf{P}) = \mathbf{G}'(\mathbf{F}')$.

In general, the response of the material relative to $\mathcal{B}'_r$ differs from that relative to $\mathcal{B}_r$, i.e $\mathbf{G}' \neq \mathbf{G}$. However, for specific $\mathbf{P}$ we may have $\mathbf{G}' = \mathbf{G}$, in which case

$$\mathbf{G}(\mathbf{F}'\mathbf{P}) = \mathbf{G}(\mathbf{F}') \tag{2.48}$$

for all deformation gradients $\mathbf{F}'$ and for all such $\mathbf{P}$. Equation (2.46) then gives $\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) = \mathbf{G}(\mathbf{F}')$, and, in order to calculate $\boldsymbol{\sigma}$, it is not necessary to distinguish between $\mathcal{B}_r$ and $\mathcal{B}'_r$.

The set of tensors $\mathbf{P}$ for which (2.48) holds forms a multiplicative group, called the *symmetry group of the material relative to* $\mathcal{B}_r$. This group characterizes the physical symmetry properties of the material.

Let $\mathbf{P}$ be the deformation gradient $\mathcal{B}_r \to \mathcal{B}'_r$, and now we do *not* assume that $\mathbf{P}$ is a member of the symmetry group. Then, if $\mathcal{G}$ is the symmetry group of the material relative to $\mathcal{B}_r$ and $\mathcal{G}'$ that relative to $\mathcal{B}'_r$ these groups are related according to *Noll's rule*

$$\mathcal{G}' = \mathbf{P}\mathcal{G}\mathbf{P}^{-1}. \tag{2.49}$$

Clearly, for the special case in which $\mathbf{P} \in \mathcal{G}$, we have $\mathcal{G}' = \mathcal{G}$.

1.2.3.3 Isotropic elasticity

If $\mathcal{G}$ is the proper orthogonal group then the material is said to be *isotropic relative to* $\mathcal{B}_r$, and then

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}\mathbf{Q}) = \mathbf{G}(\mathbf{F}) \tag{2.50}$$

for all proper orthogonal $\mathbf{Q}$ and for every deformation gradient $\mathbf{F}$. Physically, this means that the response of a small specimen of material is independent of its orientation in $\mathcal{B}_r$.

Before proceeding further we require some definitions and results relating to isotropic functions of a second-order tensor. Firstly, the scalar function $\phi(\mathbf{T})$ of a *symmetric* second-order tensor $\mathbf{T}$ is said to be an *isotropic function* of $\mathbf{T}$ if

$$\phi(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \phi(\mathbf{T}) \tag{2.51}$$

for all orthogonal tensors $\mathbf{Q}$. An isotropic scalar-valued function of $\mathbf{T}$ is also called a *scalar invariant* of $\mathbf{T}$. It may easily be checked that the *principal invariants* of $\mathbf{T}$, defined by

$$I_1(\mathbf{T}) = \mathrm{tr}\,(\mathbf{T}), \quad I_2(\mathbf{T}) = \frac{1}{2}[I_1(\mathbf{T})^2 - \mathrm{tr}\,(\mathbf{T}^2)], \quad I_3(\mathbf{T}) = \det \mathbf{T}, \tag{2.52}$$

are scalar invariants in accordance with the definition (2.51). It may be shown that $\phi(\mathbf{T})$ is a scalar invariant of $\mathbf{T}$ if and only if it is expressible as a function of $I_1(\mathbf{T}), I_2(\mathbf{T}), I_3(\mathbf{T})$.

Secondly, suppose that $\mathbf{G}(\mathbf{T})$ is a symmetric second-order tensor function of $\mathbf{T}$. Then, $\mathbf{G}(\mathbf{T})$ is said to be an *isotropic tensor function* of $\mathbf{T}$ if

$$\mathbf{G}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{G}(\mathbf{T})\mathbf{Q}^T \tag{2.53}$$

for all orthogonal $\mathbf{Q}$. Consequences of this are (i) if $\mathbf{G}(\mathbf{T})$ is isotropic then its eigenvalues are scalar invariants of $\mathbf{T}$, (ii) $\mathbf{G}(\mathbf{T})$ is coaxial with $\mathbf{T}$, i.e.

$$\mathbf{G}(\mathbf{T})\mathbf{T} = \mathbf{T}\mathbf{G}(\mathbf{T}), \tag{2.54}$$

and (iii) $\mathbf{G}(\mathbf{T})$ is isotropic if and only if it has the representation

$$\mathbf{G}(\mathbf{T}) = \phi_0\mathbf{I} + \phi_1\mathbf{T} + \phi_2\mathbf{T}^2, \tag{2.55}$$

where ϕ_0, ϕ_1, ϕ_2 are scalar invariants of $\mathbf{T}$ and hence functions of $I_1(\mathbf{T})$, $I_2(\mathbf{T})$, $I_3(\mathbf{T})$.

The choice $\mathbf{Q} = \mathbf{R}^T$ and use of the polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$ in (2.50) gives

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V}). \tag{2.56}$$

We then obtain

$$\mathbf{Q}\mathbf{G}(\mathbf{V})\mathbf{Q}^T = \mathbf{G}(\mathbf{Q}\mathbf{V}\mathbf{Q}^T) \tag{2.57}$$

for all proper orthogonal $\mathbf{Q}$. In fact, since $\mathbf{Q}$ occurs twice on each side of (2.57), allowing $\mathbf{Q}$ to be improper orthogonal does not affect (2.57), which then states that $\mathbf{G}(\mathbf{V})$ is an isotropic function of $\mathbf{V}$ in accordance with the definition (2.53).

In particular, for an *isotropic elastic material,* $\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V})$ *is coaxial with* $\mathbf{V}$, i.e. with the Eulerian principal axes, and we therefore have

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V}) = \phi_0\mathbf{I} + \phi_1\mathbf{V} + \phi_2\mathbf{V}^2, \tag{2.58}$$

where ϕ_0, ϕ_1, ϕ_2 are scalar invariants of $\mathbf{V}$, i.e. functions of

$$i_1 = I_1(\mathbf{V}) = \mathrm{tr}\,(\mathbf{V}) = \lambda_1 + \lambda_2 + \lambda_3, \tag{2.59}$$

$$i_2 = I_2(\mathbf{V}) = \frac{1}{2}[i_1^2 - \mathrm{tr}\,(\mathbf{V}^2)] = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \tag{2.60}$$

$$i_3 = I_3(\mathbf{V}) = \det\mathbf{V} = \lambda_1\lambda_2\lambda_3, \tag{2.61}$$

where the expressions have also been given in terms of the principal stretches and the notation i_1, i_2, i_3 has been introduced specifically for the principal invariants of $\mathbf{V}$ (and hence of $\mathbf{U}$). Alternatively, we may write

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.62}$$

where

$$\sigma_i = \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2 \qquad i \in \{1,2,3\}, \tag{2.63}$$

and this allows us to introduce the *scalar response function* g, such that

$$\sigma_i = g(\lambda_i, \lambda_j, \lambda_k) = g(\lambda_i, \lambda_k, \lambda_j) \equiv \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2, \tag{2.64}$$

where (i, j, k) is permutation of $(1, 2, 3)$.

The expansion (2.58) may be written, equivalently, in terms of $\mathbf{B} = \mathbf{V}^2$. For example,

$$\boldsymbol{\sigma} = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2, \tag{2.65}$$

or

$$\boldsymbol{\sigma} = \beta_0\mathbf{I} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1}, \tag{2.66}$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_{-1}$ are scalar invariants of $\mathbf{B}$ (and hence of $\mathbf{V}$); see, for example, Beatty (1987). Connections between these different coefficients are determined by using the Cayley-Hamilton theorem in the form

$$\mathbf{V}^3 - i_1\mathbf{V}^2 + i_2\mathbf{V} - i_3\mathbf{I} = \mathbf{O} \tag{2.67}$$

or its counterpart for $\mathbf{B}$. It is convenient in what follows to use the standard notation I_1, I_2, I_3 for the principal invariants of $\mathbf{B}$ (also of $\mathbf{C}$). Thus, specifically, we write

$$I_1 = I_1(\mathbf{B}) = \mathrm{tr}\,(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.68}$$

$$I_2 = I_2(\mathbf{B}) = \frac{1}{2}[I_1^2 - \mathrm{tr}\,(\mathbf{B}^2)] = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{2.69}$$

$$I_3 = I_3(\mathbf{B}) = \det \mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.70}$$

In view of the connection (2.28) between $\mathbf{S}$ and $\boldsymbol{\sigma}$ we may also define the response function, $\mathbf{H}$ say, associated with $\mathbf{S}$ (relative to $\mathcal{B}_r$) by

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) \equiv J\mathbf{F}^{-1}\mathbf{G}(\mathbf{F}). \tag{2.71}$$

The objectivity requirement (2.44) then becomes

$$\mathbf{H}(\mathbf{QF}) = \mathbf{H}(\mathbf{F})\mathbf{Q}^T. \tag{2.72}$$

A corresponding change for the material symmetry transformation (2.48) can be written down, and, in particular, for an isotropic elastic solid, we have

$$\mathbf{H}(\mathbf{FQ}) = \mathbf{Q}^T\mathbf{H}(\mathbf{F}). \tag{2.73}$$

Moreover, it follows from (2.73) that

$$\mathbf{H}(\mathbf{F}) = \mathbf{H}(\mathbf{U})\mathbf{R}^T = \mathbf{R}^T\mathbf{H}(\mathbf{V}), \tag{2.74}$$

with $\mathbf{H}(\mathbf{U})$ being symmetric and coaxial with $\mathbf{U}$.

1.2.3.4 Internal constraints

In Section 1.2.1 the (internal) constraints of incompressibility and inextensibility were mentioned. More generally, a single constraint may be written in the form

$$C(\mathbf{F}) = 0, \tag{2.75}$$

where C is a scalar function. Equation (2.75) holds for all possible deformation gradients $\mathbf{F}$. For the incompressibility and inextensibility constraints we have, respectively,

$$C(\mathbf{F}) = \det \mathbf{F} - 1, \qquad C(\mathbf{F}) = \mathbf{M} \cdot (\mathbf{F}^T\mathbf{F}\mathbf{M}) - 1. \tag{2.76}$$

Since any constraint is unaffected by a superimposed rigid deformation, C must be an objective scalar function, so that

$$C(\mathbf{Q}\mathbf{F}) = C(\mathbf{F}) \tag{2.77}$$

for all rotations $\mathbf{Q}$. In particular, the choice $\mathbf{Q} = \mathbf{R}^T$ yields

$$C(\mathbf{F}) = C(\mathbf{U}). \tag{2.78}$$

For incompressibility the $C(\mathbf{U})$ given by $(2.76)_1$ is a scalar invariant of $\mathbf{U}$, but this is not the case for a general constraint function $C(\mathbf{U})$.

The constraint (2.75) defines a hypersurface in the (nine-dimensional) space of deformation gradients. Any stress in the normal direction to the surface (i.e. the direction $\partial C/\partial \mathbf{F}$) does no work in any (virtual) incremental deformation $\delta\mathbf{x}$ compatible with the constraint since $\mathrm{tr}\,[(\partial C/\partial \mathbf{F})\delta\mathbf{F}] = 0$. The stress is therefore determined by the constitutive law, in the form (2.71) for example, only to within an additive contribution parallel to the normal. Thus, for a constrained material the stress-deformation relation (2.71) is replaced by

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) + q\frac{\partial C}{\partial \mathbf{F}}, \tag{2.79}$$

or, in terms of Cauchy stress,

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) + qJ^{-1}\mathbf{F}\frac{\partial C}{\partial \mathbf{F}}, \tag{2.80}$$

where q is an arbitrary (Lagrange) multiplier. The term in q is referred to as the *constraint stress* since it arises from the constraint and is not otherwise derivable from the material properties.

For incompressibility we have $\partial C/\partial \mathbf{F} = \mathbf{F}^{-1}$ since $J = 1$ and hence

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) + q\mathbf{F}^{-1}, \qquad \boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) + q\mathbf{I}, \tag{2.81}$$

$\mathbf{I}$ again being the identity tensor, while for inextensibility $\partial C/\partial \mathbf{F} = 2\mathbf{M} \otimes \mathbf{FM}$ and

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) + 2q\mathbf{M} \otimes \mathbf{FM}, \qquad \boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) + 2qJ^{-1}\mathbf{FM} \otimes \mathbf{FM}. \tag{2.82}$$

In the case of (2.81), if the material is isotropic then $\mathbf{G}(\mathbf{F})$ is given by (2.58) but with the term in ϕ_0 omitted since it may be absorbed into q and ϕ_1 and ϕ_2 being functions of the two remaining independent invariants.

Another constraint, called the *Bell constraint*, is the focus of Chapter 2 and will not therefore be discussed here. If there is more than one constraint then an additive constraint stress has to be included in the expression for the stress in respect of each constraint. However, the constraints must be mutually compatible since, as illustrated in Chapter 2, the incompressibility and Bell constraints are not compatible.

1.2.4 Hyperelasticity

As mentioned at the beginning of Section 1.2.3, the notion of elasticity introduced there is referred to as *Cauchy elasticity*. From the point of view of both theory and applications a more useful concept of elasticity, which is a special case of Cauchy elasticity, is *hyperelasticity* (or *Green elasticity*). In this theory there exists a *strain-energy function* (or *stored-energy function*), denoted $W = W(\mathbf{F})$, defined on the space of deformation gradients such that (for an unconstrained material)

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}, \quad \boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{2.83}$$

The work increment in (2.35) is then converted into stored energy and is simply $\mathrm{tr}\,(\mathbf{S}\delta\mathbf{F}) = \delta W$. Equation (2.83) is the *stress-deformation relation* or *constitutive relation* for an elastic material which possesses a strain-energy function, W being defined per unit volume in $\mathcal{B}_r$ and representing the work done per unit volume at $\mathbf{X}$ in changing the deformation gradient from $\mathbf{I}$ to $\mathbf{F}$. In components, the first equation in (2.83) is written $S_{\alpha i} = \partial W/\partial F_{i\alpha}$, which provides the convention for ordering of the indices in the partial derivative with respect to $\mathbf{F}$.

Henceforth in this chapter we restrict attention to hyperelasticity and regard an elastic material as characterized by the existence of a strain-energy function such that (2.83) hold. We take W and the stress to vanish in $\mathcal{B}_r$, so that

$$W(\mathbf{I}) = 0, \qquad \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{O}, \tag{2.84}$$

the latter being consistent with (2.40).

We note here the modification of (2.83) appropriate for incompressibility. From (2.81) we obtain

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{F}^{-1}, \quad \boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \quad \det \mathbf{F} = 1, \tag{2.85}$$

where q has been replaced by $-p$, with p, in standard notation, then referred to as an *arbitrary hydrostatic pressure.* Equations (2.85) are *stress-deformation relations for an incompressible elastic material.* The corresponding expressions for the constraint of inextensibility may be read off from (2.82).

1.2.4.1 Objectivity and Material Symmetry

The elastic stored energy is required to be independent of superimposed rigid deformations of the form (2.41) and it therefore follows that

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{2.86}$$

for *all* rotations $\mathbf{Q}$. A strain-energy function satisfying this requirement is said to be *objective.*

Use of the polar decomposition (2.17) and the choice $\mathbf{Q} = \mathbf{R}^T$ in (2.86) shows that

$$W(\mathbf{F}) = W(\mathbf{U}). \tag{2.87}$$

Thus, W depends on $\mathbf{F}$ only through the stretch tensor $\mathbf{U}$ and may therefore be defined on the class of positive definite symmetric tensors. Since $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$, as defined in (2.23), is conjugate to the Biot stress tensor $\mathbf{T}^{(1)}$, which we write henceforth as $\mathbf{T}$, we have

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} \tag{2.88}$$

for an unconstrained material and

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} - p\mathbf{U}^{-1}, \qquad \det \mathbf{U} = 1 \tag{2.89}$$

for an incompressible material. Note that when expressed as a function of $\mathbf{U}$ the strain energy automatically satisfies the objectivity requirement.

Mathematically, there is no restriction so far other than (2.84) and (2.86) on the form that the function W may take, but the predicted stress-strain behaviour based on the form of W must on the one hand be acceptable for the description of the elastic behaviour of real materials and on the other hand make mathematical sense.

Further restrictions on the form of W arise if the material possesses symmetries in the configuration $\mathcal{B}_r$. For a hyperelastic material the symmetry requirement (2.48) is replaced by

$$W(\mathbf{F}'\mathbf{P}) = W(\mathbf{F}') \tag{2.90}$$

for *all* deformation gradients $\mathbf{F}'$. This states that the strain-energy function is unaffected by a change of reference configuration with deformation gradient $\mathbf{P}$ which is a member of the symmetry group of the material relative to $\mathcal{B}_r$.

1.2.4.2 Isotropic hyperelasticity

To be specific we now consider *isotropic elastic materials*, for which the symmetry group is the *proper orthogonal group*. Then, we have

$$W(\mathbf{FQ}) = W(\mathbf{F}) \tag{2.91}$$

for *all* rotations $\mathbf{Q}$. Bearing in mind that the $\mathbf{Q}$'s appearing in (2.86) and (2.91) are independent the combination of these two equations yields

$$W(\mathbf{QUQ}^T) = W(\mathbf{U}) \tag{2.92}$$

for all rotations $\mathbf{Q}$, or, equivalently, $W(\mathbf{QVQ}^T) = W(\mathbf{V})$. Equation (2.92) states that W is an *isotropic function* of $\mathbf{U}$. It follows from the spectral decomposition (2.18) that W depends on $\mathbf{U}$ only through the principal stretches $\lambda_1, \lambda_2, \lambda_3$. To avoid introducing additional notation we express this dependence as $W(\lambda_1, \lambda_2, \lambda_3)$; by selecting appropriate values for $\mathbf{Q}$ in (2.92) we may deduce that W depends symmetrically on $\lambda_1, \lambda_2, \lambda_3$, i.e.

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_2, \lambda_1, \lambda_3). \tag{2.93}$$

A consequence of isotropy is that $\mathbf{T}$ is *coaxial* with $\mathbf{U}$ and hence, in parallel with (2.18), we have

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.94}$$

where t_i, $i \in \{1, 2, 3\}$ are the *principal Biot stresses*. For an unconstrained material,

$$t_i = \frac{\partial W}{\partial \lambda_i}, \tag{2.95}$$

and for an incompressible material this is replaced by

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1}, \qquad \lambda_1\lambda_2\lambda_3 = 1. \tag{2.96}$$

For later reference, we note here that the principal Cauchy stresses σ_i, $i \in \{1,2,3\}$, are given by

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} \tag{2.97}$$

and

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \qquad \lambda_1\lambda_2\lambda_3 = 1 \tag{2.98}$$

for unconstrained and incompressible materials respectively. Note that in (2.97) and (2.98) there is no summation over the repeated index i.

With reference to (2.63) it can be deduced from (2.97), by regarding W as a function of i_1, i_2, i_3, that the coefficients ϕ_0, ϕ_1, ϕ_2 are given by

$$\phi_0 = \frac{\partial W}{\partial i_3}, \quad \phi_1 = i_3^{-1}\Big(\frac{\partial W}{\partial i_1} + i_1\frac{\partial W}{\partial i_2}\Big), \quad \phi_2 = -i_3^{-1}\frac{\partial W}{\partial i_2}. \tag{2.99}$$

Similarly, the coefficients $\alpha_0, \alpha_1, \alpha_2$ in (2.65) are given by

$$\alpha_0 = 2I_3^{1/2}\frac{\partial W}{\partial I_3}, \quad \alpha_1 = 2I_3^{-1/2}\Big(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\Big), \quad \alpha_2 = -2I_3^{-1/2}\frac{\partial W}{\partial I_2}, \tag{2.100}$$

where W is now regarded as a function of I_1, I_2, I_3. For an incompressible material the term in ϕ_0, or α_0 as appropriate, is absorbed into p and $i_3 = I_3 = 1$ in the remaining terms in (2.99) and (2.100).

We emphasize that, as follows from (2.74), *for an isotropic elastic material* **SR** *is symmetric* and we have

$$\mathbf{S} = \mathbf{T}\mathbf{R}^T = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.101}$$

The first equation in (2.101) is a polar decomposition of **S** analogous to $(2.17)_1$ except that it is not unique. This equation has important consequences for considerations of uniqueness and stability and will be discussed briefly in Section 1.4.4.

1.2.4.3 Examples of strain-energy functions

There are numerous specific forms of strain-energy function in the literature both for compressible and incompressible materials, mainly isotropic, and we make no attempt to catalogue them here. Many will be used in the various chapters of this volume. Here we just mention, for purposes of illustration, a few examples of strain-energy functions for *incompressible isotropic* materials. Many, but not all, of the compressible strain-energy functions used in the literature are obtained from their incompressible counterparts by addition of a function of the volume measure J and multiplication of the other terms by powers of J. Examples of compressible strain-energy functions are contained in Chapter 4, for example.

As already noted the strain energy of an isotropic elastic solid can be regarded either as a symmetric function of the principal stretches or as a function of three independent invariants, such as i_1, i_2, i_3 or I_1, I_2, I_3. For an incompressible material (2.10) holds, $i_3 = I_3 = 1$ and hence the strain energy depends on only two independent invariants. An important example is the *Mooney-Rivlin* form of strain energy, defined by

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \equiv C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3), \tag{2.102}$$

where C_1, C_2 are constants and $\lambda_1\lambda_2\lambda_3 = 1$. When $C_2 = 0$ this reduces to the so-called *neo-Hookean* strain energy

$$W = \frac{1}{2}\mu(I_1 - 3) \equiv \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \tag{2.103}$$

where C_1 has been replaced by $\mu/2$, $\mu\,(> 0)$ being the shear modulus of the material in the undeformed configuration. These two forms of energy function played key roles in the development of the subject of finite elasticity, particularly in respect of its connection with rubber elasticity. For reviews of this aspect we refer to Ogden (1982, 1986) in which more details are given of different forms of strain-energy functions appropriate for rubberlike solids. Equation (2.102) constitutes the linear terms in a polynomial expansion of W in terms of $I_1 - 3$ and $I_2 - 3$, special cases of which are used extensively in the literature.

Another special form of strain energy worthy of mention is the Varga form, defined by

$$W = 2\mu(i_1 - 3) \equiv 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{2.104}$$

which is used extensively in basic stress-strain analysis (Varga, 1966). See also Chapter 5 in this volume.

Each of the strain-energy functions (2.102)–(2.104) is of the *separable form*

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3), \tag{2.105}$$

which was introduced by Valanis and Landel (1967). Equivalent to (2.105) is the expansion

$$W = \sum_{m=1}^{\infty} \mu_m(\lambda_1^{\alpha_m} + \lambda_2^{\alpha_m} + \lambda_3^{\alpha_m} - 3)/\alpha_m \tag{2.106}$$

in terms of powers of the stretches, where each μ_m and α_m is a material constant, the latter not necessarily being integers (Ogden, 1972, 1982). For practical purposes the sum in (2.106) is restricted to a finite number of terms, while, for consistency with the classical theory, the constants must satisfy the requirement

$$\sum_{m=1}^{N} \mu_m\alpha_m = 2\mu, \tag{2.107}$$

where N is a positive integer and μ is again the shear modulus of the material in the natural configuration. The counterpart of (2.107) for the Valanis-Landel material is

$$w''(1) + w'(1) = 2\mu. \tag{2.108}$$

In respect of (2.105) the principal Cauchy stresses are obtained from (2.98) in the form

$$\sigma_i = \lambda_i w'(\lambda_i) - p \tag{2.109}$$

and the specializations appropriate for (2.102)–(2.104) are then easily read off. For the neo-Hookean solid, for example, we have

$$\boldsymbol{\sigma} = \mu\mathbf{B} - p\mathbf{I}, \tag{2.110}$$

where $\mathbf{B}$ is the left Cauchy-Green deformation tensor defined in (2.22).

1.2.4.4 Anisotropy: fibre-reinforced materials

For a general discussion of anisotropic elasticity, including the crystal classes, we refer to Green and Adkins (1970) or Truesdell and Noll (1965), for example. Here, we illustrate the structure of the strain-energy function of an anisotropic elastic solid for the example of *transverse isotropy*, in which there is a single preferred direction, and the extension of this to the case of two preferred directions. These are important examples in practical applications to fibre-reinforced materials, such as high-pressure hoses and soft biological tissues.

Firstly, we consider transverse isotropy. Let the unit vector $\mathbf{M}$ be a preferred direction in the reference configuration of the material. The material response is then indifferent to arbitrary rotations about the direction $\mathbf{M}$ and by replacement of $\mathbf{M}$ by $-\mathbf{M}$. Such a material can be characterized with a strain energy which depends on $\mathbf{F}$ and the tensor $\mathbf{M}\otimes\mathbf{M}$, as described by Spencer (1972, 1984); see, also, Rogers (1984a) and Holzapfel (2000). Thus, we write $W(\mathbf{F},\mathbf{M}\otimes\mathbf{M})$ and the required symmetry reduces W to dependence on the five invariants

$$I_1,\ I_2,\ I_3,\ I_4=\mathbf{M}\cdot(\mathbf{CM}),\ I_5=\mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}), \tag{2.111}$$

where I_1, I_2, I_3 are defined in (2.68)–(2.70). The resulting nominal stress tensor is given by

$$\begin{aligned}\mathbf{S}=2W_1\mathbf{F}^T+2W_2(I_1\mathbf{I}-\mathbf{C})\mathbf{F}^T+2I_3W_3\mathbf{F}^{-1}+2W_4\mathbf{M}\otimes\mathbf{FM}\\+2W_5(\mathbf{M}\otimes\mathbf{FCM}+\mathbf{CM}\otimes\mathbf{FM}),\end{aligned} \tag{2.112}$$

where $W_i=\partial W/\partial I_i, i=1,\dots,5$. For an isotropic material the terms in W_4 and W_5 are omitted. Equation (2.112) describes the response of a fibre-reinforced material with the fibre direction corresponding to $\mathbf{M}$ locally in the reference configuration. For an incompressible material the dependence on I_3 is omitted and the Cauchy stress tensor is given by

$$\begin{aligned}\boldsymbol{\sigma}=-p\mathbf{I}+2W_1\mathbf{B}+2W_2(I_1\mathbf{B}-\mathbf{B}^2)+2W_4\mathbf{FM}\otimes\mathbf{FM}\\+2W_5(\mathbf{FM}\otimes\mathbf{BFM}+\mathbf{BFM}\otimes\mathbf{FM}),\end{aligned} \tag{2.113}$$

from which the symmetry of $\boldsymbol{\sigma}$ can be seen immediately. Note that in (2.113) the left Cauchy-Green tensor $\mathbf{B}$ has been used.

When there are two families of fibres corresponding to two preferred directions in the reference configuration, $\mathbf{M}$ and $\mathbf{M}'$ say, then, in addition to (2.111), the strain energy depends on the invariants

$$I_6=\mathbf{M}'\cdot(\mathbf{CM}'),\ I_7=\mathbf{M}'\cdot(\mathbf{C}^2\mathbf{M}'),\ I_8=\mathbf{M}\cdot(\mathbf{CM}'), \tag{2.114}$$

and also on $\mathbf{M}\cdot\mathbf{M}'$ (which does not depend on the deformation); see Spencer (1972, 1984) for details. Note that I_8 involves interaction between the two preferred directions, but the term $\mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}')$, which might be expected to appear in the list (2.114), is omitted since it depends on the other invariants and on $\mathbf{M}\cdot\mathbf{M}'$. It suffices here to give the Cauchy stress tensor for an incompressible material. This is

$$\begin{aligned}\boldsymbol{\sigma}=-p\mathbf{I}+2W_1\mathbf{B}+2W_2(I_1\mathbf{B}-\mathbf{B}^2)+2W_4\mathbf{FM}\otimes\mathbf{FM}\\+2W_5(\mathbf{FM}\otimes\mathbf{BFM}+\mathbf{BFM}\otimes\mathbf{FM})+2W_6\mathbf{FM}'\otimes\mathbf{FM}'\\+2W_7(\mathbf{FM}'\otimes\mathbf{BFM}'+\mathbf{BFM}'\otimes\mathbf{FM}')\\+W_8(\mathbf{FM}\otimes\mathbf{FM}'+\mathbf{FM}'\otimes\mathbf{FM}),\end{aligned} \tag{2.115}$$

where the notation $W_i=\partial W/\partial I_i$ now applies for $i=1,\dots,8$. In the context of finite deformation theory very few boundary-value problems have been solved for either transversely isotropic materials or for materials with two preferred directions. An account of such results is given in Green and Adkins (1970), but relatively little progress has been made since the publication of this book as far as obtaining closed-form solutions is concerned. For materials with one or two families of *inextensible* fibres some basic results are contained in Spencer (1972, 1984) and Rogers (1984b). However, there is very little in the literature concerned with *specific* forms of strain-energy function based on the invariants (2.111) or (2.111) combined with (2.114), although, in the context of membrane biomechanics, special forms of (2.113) have been used; see, for example, Humphrey (1995) for references.

1.2.5 Boundary-value problems

We now consider the equilibrium equation (2.31) together with the stress-deformation relation $(2.83)_1$ for an unconstrained material, and the deformation gradient (2.3) coupled with (2.1). Thus,

$$\mathrm{Div}\left(\frac{\partial W}{\partial \mathbf{F}}\right) + \rho_r \mathbf{b} = \mathbf{0}, \quad \mathbf{F} = \mathrm{Grad}\,\mathbf{x}, \quad \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \qquad \mathbf{X} \in \mathcal{B}_r. \tag{2.116}$$

A boundary-value problem is obtained by supplementing (2.116) with appropriate boundary conditions. Typical boundary conditions arising in problems of nonlinear elasticity are those in which $\mathbf{x}$ is specified on part of the boundary, $\partial\mathcal{B}_r^x \subset \partial\mathcal{B}_r$ say, and the stress vector on the remainder, $\partial\mathcal{B}_r^\tau$, so that $\partial\mathcal{B}_r^x \cup \partial\mathcal{B}_r^\tau = \partial\mathcal{B}_r$ and $\partial\mathcal{B}_r^x \cap \partial\mathcal{B}_r^\tau = \emptyset$. We write

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \qquad \text{on } \partial\mathcal{B}_r^x, \tag{2.117}$$

$$\mathbf{S}^T\mathbf{N} = \boldsymbol{\tau}(\mathbf{F}, \mathbf{X}) \qquad \text{on } \partial\mathcal{B}_r^\tau, \tag{2.118}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$ are specified functions. In general, $\boldsymbol{\tau}$ may depend on the deformation and we have indicated this in (2.118) by showing the dependence of $\boldsymbol{\tau}$ on the deformation gradient $\mathbf{F}$. If the surface traction defined by (2.118) is independent of $\mathbf{F}$ it is referred to as a *dead-load traction.* If the boundary traction in (2.118) is associated with a hydrostatic pressure, P say, so that $\boldsymbol{\sigma}\mathbf{n} = -P\mathbf{n}$, then $\boldsymbol{\tau}$ depends on the deformation in the form

$$\boldsymbol{\tau} = -JP\mathbf{F}^{-T}\mathbf{N} \quad \text{on } \partial\mathcal{B}_r^\tau. \tag{2.119}$$

In general, when the dependence on $\mathbf{F}$ is retained, (2.118) is referred to as a *configuration dependent loading* (Sewell, 1967). The basic boundary-value problem of nonlinear elasticity is characterized by (2.116)–(2.118).

In components, the equilibrium equation in (2.116) can be written

$$\mathcal{A}^1_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta} + \rho_r b_i = 0, \tag{2.120}$$

for $i \in \{1,2,3\}$, where the coefficients $\mathcal{A}^1_{\alpha i\beta j}$ are defined by

$$\mathcal{A}^1_{\alpha i\beta j} = \mathcal{A}^1_{\beta j\alpha i} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}, \tag{2.121}$$

the pairwise symmetry of the indices thereby being emphasized.

When coupled with suitable boundary conditions, equation (2.120) forms a coupled system of three second-order quasi-linear partial differential equations for $x_i = \chi_i(X_\alpha)$. The coefficients $\mathcal{A}^1_{\alpha i\beta j}$ are, in general, nonlinear functions of the components of the deformation gradient. We emphasize that here we are using Cartesian coordinates; expressions for the equilibrium equations in other coordinate systems are not given here but can be found in Ogden (1997), for example.

For *unconstrained materials* very few explicit solutions have been obtained for boundary-value problems, and these arise for very special choices of the form of W and for relatively simple geometries. References are given in Ogden (1997), for example, and an up-to-date account is contained in Chapter 4. For *incompressible materials* the corresponding equations, obtained by substituting $(2.85)_1$ into (2.31) to give

$$\mathcal{A}^1_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta} - \frac{\partial p}{\partial x_i} + \rho_r b_i = 0 \tag{2.122}$$

subject to (2.10), where the coefficients are again given by (2.121). These have yielded more success, and we refer to Green and Zerna (1968), Green and Adkins (1970) and Ogden (1997) for details of the solutions, many of which are based on the pioneering work of Rivlin (see, for example, Rivlin, 1948a, b, 1949a, b; references to further work by Rivlin and co-workers can be found in, for example, Truesdell and Noll, 1965 and Green and Zerna, 1968; the edited papers of Rivlin are provided in the volume by Barenblatt and Joseph, 1996). See also Chapters 3, 4, and 5 in this volume for further discussion of boundary-value problems. Note that, exceptionally, for the neo-Hookean form of strain energy (2.103), the coefficients $\mathcal{A}^1_{\alpha i\beta j}$ are constant and given by

$$\mathcal{A}^1_{\alpha i\beta j} = \mu\delta_{ij}\delta_{\alpha\beta}. \tag{2.123}$$

The equations (2.122), although appearing linear in this case, are in general nonlinear because of the term in p.

In order to analyze such boundary-value problems additional information about the nature of the function W is required. This information may come from the construction of special forms of strain-energy function based on comparison of theory with experiment for particular materials, may arise naturally in the course of solution of particular problems, or may be derived from mathematical considerations relating to the properties that W should possess in order for existence of solutions to be guaranteed, for example. This important aspect of the theory is not examined to any great extent in this chapter, but reference can be made to, for example, Ciarlet (1988) or Ogden (1997) for discussion of this matter. In this connection, however, we now examine certain aspects of the structure of the equations.

Equations (2.120) are said to be *strongly elliptic* if the inequality

$$\mathcal{A}^1_{\alpha i\beta j} m_i m_j N_\alpha N_\beta > 0 \tag{2.124}$$

holds for all non-zero vectors $\mathbf{m}$ and $\mathbf{N}$. Note that this inequality is independent of any boundary conditions. Strong ellipticity ensures, in particular, that in an infinite medium infinitesimal motions superimposed on a finite deformation do not grow exponentially (see Section 1.6.3.1).

For an incompressible material the strong ellipticity condition associated with (2.122) again has the form (2.124) but the incompressibility constraint now imposes the restriction

$$\mathbf{m} \cdot \mathbf{n} = 0 \tag{2.125}$$

on the (non-zero) vectors $\mathbf{m}$ and $\mathbf{n}$, where $\mathbf{n}$, the push forward of $\mathbf{N}$, is defined by

$$\mathbf{F}^T \mathbf{n} = \mathbf{N}. \tag{2.126}$$

Note that $\mathbf{N}$ and $\mathbf{n}$ here are not related to the surface normal vectors defined in Section 1.2.1.

In terms of $\mathbf{n}$ the strong ellipticity condition (2.124) may be written

$$\mathcal{A}^1_{0piqj} m_i m_j n_p n_q > 0, \tag{2.127}$$

where $\mathcal{A}^1_{0piqj}$ are related to $\mathcal{A}^1_{\alpha i\beta j}$ by

$$\mathcal{A}^1_{0piqj} = J^{-1} F_{p\alpha} F_{q\beta} \mathcal{A}^1_{\alpha i\beta j}. \tag{2.128}$$

For an isotropic material necessary and sufficient conditions for strong ellipticity to hold *in two dimensions* for unconstrained and incompressible materials will be given in Section 1.4.2.2. The corresponding conditions for three dimensions are quite complicated and are not therefore given here. We refer to Zee and Sternberg (1983) for incompressible materials and Simpson and Spector

(1983), Rosakis (1990) and Wang and Aron (1996) for unconstrained materials. See also Hill (1979).

Failure of ellipticity in the sense that equality holds in (2.127) for some specific $\mathbf{m}$ and $\mathbf{n}$ plays a very important role in connection with the emergence of solutions with discontinuous deformation gradients. For discussion of this see, for example, Knowles and Sternberg (1975, 1977) and Chapter 12 in this volume, which contains more detailed references to work on this topic.

There are some important situations where simplifications of the governing equations (2.120) or (2.122) arise. For example, for *plane strain deformations*, equation (2.120) reduces to a pair of equations for $x_\alpha = \chi_\alpha(X_\beta)$, where the Greek indices have the range $\{1,2\}$. A further simplification arises for *anti-plane shear deformations*, for which (2.120) reduces to a single quasi-linear equation for a scalar function $x = \chi(X_\alpha)$, $\alpha \in \{1,2\}$. For a review of anti-plane shear see, for example, Horgan (1995).

1.2.5.1 Variational structure

We now take the body force in (2.34) to be conservative so that we may write

$$\mathbf{b} = -\mathrm{grad}\,\phi, \tag{2.129}$$

where ϕ is a scalar field defined on points in $\mathcal{B}$ and grad denotes the gradient operator in $\mathcal{B}$. Since we are considering an elastic material with strain-energy function W, it follows that the virtual work equation (2.34) can be expressed in the form

$$\delta\int_{\mathcal{B}_r}(W+\rho_r\phi)\,\mathrm{d}V - \int_{\partial\mathcal{B}_r}(\mathbf{S}^T\mathbf{N})\cdot\delta\mathbf{x}\,\mathrm{d}A = 0. \tag{2.130}$$

In view of the boundary condition (2.117) we have $\delta\mathbf{x} = \mathbf{0}$ on $\partial\mathcal{B}_r^x$, and (2.130) becomes

$$\delta\int_{\mathcal{B}_r}(W+\rho_r\phi)\,\mathrm{d}V - \int_{\partial\mathcal{B}_r^\tau}\boldsymbol{\tau}\cdot\delta\mathbf{x}\,\mathrm{d}A = 0. \tag{2.131}$$

For illustrative purposes and for simplicity we now take $\boldsymbol{\tau}$ to be independent of the deformation and we write $\boldsymbol{\tau} = \mathrm{grad}\,(\boldsymbol{\tau}\cdot\mathbf{x})$, so that (2.131) becomes

$$\delta\Big\{\int_{\mathcal{B}_r}(W+\rho_r\phi)\,\mathrm{d}V - \int_{\partial\mathcal{B}_r^\tau}\boldsymbol{\tau}\cdot\mathbf{x}\,\mathrm{d}A\Big\} = 0. \tag{2.132}$$

If we regard $\delta\boldsymbol{\chi}$ as a variation of the function $\boldsymbol{\chi}$, then (2.132) provides a variational formulation of the boundary-value problem (2.116)–(2.118) and can be written $\delta E = 0$, where E is the *functional* defined by

$$E\{\boldsymbol{\chi}\} = \int_{\mathcal{B}_r}\{W(\mathrm{Grad}\,\boldsymbol{\chi}) + \rho_r\phi(\boldsymbol{\chi})\}\,\mathrm{d}V - \int_{\partial\mathcal{B}_r^\tau}\boldsymbol{\tau}\cdot\boldsymbol{\chi}\,\mathrm{d}A. \tag{2.133}$$

In (2.133) χ is taken in some appropriate class of mappings (for example, the twice continuously differentiable mappings we have agreed to consider in this chapter) and to satisfy the boundary condition (2.117). Similarly for the admissible variations $\delta\chi$ subject to $\delta\chi = \mathbf{0}$ on $\partial\mathcal{B}_r^x$.

Then, starting from (2.133) with χ in this class, we obtain, after use of the divergence theorem and the boundary conditions on $\partial\mathcal{B}_r^x$,

$$\delta E = -\int_{\mathcal{B}_r} (\operatorname{Div}\mathbf{S} + \rho_r \mathbf{b}) \cdot \delta\chi \, \mathrm{d}V + \int_{\partial\mathcal{B}_r^\tau} (\mathbf{S}^T\mathbf{N} - \boldsymbol{\tau}) \cdot \delta\chi \, \mathrm{d}A.$$

The variational statement then takes the form $\delta E = 0$ for all admissible $\delta\chi$ if and only if χ is such that, with $\mathbf{F} = \operatorname{Grad}\chi$ and $\mathbf{S}$ given by $(2.83)_1$ (for the unconstrained case), the equilibrium equations (2.31) and the boundary conditions (2.118) are satisfied. In other words, E is stationary if and only if χ is an actual solution (not necessarily unique) of the boundary-value problem.

As will be discussed in Section 1.4.3, the energy functional plays an important role in the analysis of stability for the dead-load problem. Detailed discussion of variational principles, including stationary energy, complementary energy and mixed principles in the context of nonlinear elasticity is contained in Ogden (1997); technical mathematical aspects are discussed in Ciarlet (1988), Marsden and Hughes (1983) and the paper by Ball (1977), for example.

1.3 Examples of boundary-value problems

1.3.1 Homogeneous deformations

1.3.1.1 Isotropic materials

We consider first some elementary problems in which the deformation is *homogeneous*, that is for which the deformation gradient $\mathbf{F}$ is constant.

A *pure homogeneous strain* is a deformation of the form

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, and, since the deformation is homogeneous, they are constants. For this deformation $\mathbf{F} = \mathbf{U} = \mathbf{V}, \mathbf{R} = \mathbf{I}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions and are fixed as the values of the stretches change. For an unconstrained isotropic elastic material the associated principal Biot stresses are given by (2.95). These equations serve as a basis for determining the form of W from triaxial experimental tests in which $\lambda_1, \lambda_2, \lambda_3$ and t_1, t_2, t_3 are measured. If biaxial tests are conducted on a thin sheet of material which lies in the (X_1, X_2)-plane with no force applied to the faces of the sheet then equations

(2.95) reduce to

$$t_1 = \frac{\partial W}{\partial \lambda_1}(\lambda_1, \lambda_2, \lambda_3), \quad t_2 = \frac{\partial W}{\partial \lambda_2}(\lambda_1, \lambda_2, \lambda_3), \quad t_3 = \frac{\partial W}{\partial \lambda_3}(\lambda_1, \lambda_2, \lambda_3) = 0, \tag{3.2}$$

and the third equation gives λ_3 implicitly in terms of λ_1 and λ_2 when W is known.

The biaxial test is more important in the context of the incompressibility constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{3.3}$$

since then only two stretches can be varied independently and biaxial tests are sufficient to obtain a characterization of W. The counterpart of (2.95) for the incompressible case is given by (2.96), or, in terms of the principal Cauchy stresses, (2.98). It is convenient to make use of (3.3) to express the strain energy as a function of two independent stretches and for this purpose we define

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}). \tag{3.4}$$

This enables p to be eliminated from equation (2.98) and leads to

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{3.5}$$

It is important to note that, because of the incompressibility constraint, equation (3.5) is unaffected by the superposition of an arbitrary hydrostatic stress. Thus, without loss of generality, we may set $\sigma_3 = 0$ in (3.5). In terms of the principal Biot stresses we then have simply

$$t_1 = \frac{\partial \hat{W}}{\partial \lambda_1}, \quad t_2 = \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{3.6}$$

which provides two equations relating λ_1, λ_2 and t_1, t_2 and therefore a basis for characterizing $\hat{W}$ from measured biaxial data.

There are several special cases of the biaxial test which are of interest, but we just give the details for *simple tension*, for which we set $t_2 = 0$. By symmetry, the incompressibility constraint then yields $\lambda_2 = \lambda_3 = \lambda_1^{-1/2}$. The strain energy may now be treated as a function of just λ_1, and we write

$$\tilde{W}(\lambda_1) = \hat{W}(\lambda_1, \lambda_1^{-1/2}), \tag{3.7}$$

and (3.6) reduces to

$$t_1 = \tilde{W}'(\lambda_1), \tag{3.8}$$

where the prime indicates differentiation with respect to λ_1.

Next we consider the *simple shear* deformation defined by

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{3.9}$$

where γ is a constant, called the *amount of shear.* This is a plane strain deformation and it provides an illustration of a deformation in which the orientation of the principal axes of strain varies with the magnitude of the deformation (in this case with γ).

For an incompressible elastic solid the Cauchy stress tensor is given by the second equation in (2.85). If this is expressed in terms of the invariants I_1, I_2 defined by (2.68)–(2.69) then, for an isotropic material, the components of $\boldsymbol{\sigma}$ are

$$\begin{aligned} \sigma_{11} &= -p + 2(1+\gamma^2)W_1 + 2(2+\gamma^2)W_2, \\ \sigma_{22} &= -p + 2W_1 + 4W_2, \quad \sigma_{12} = 2\gamma(W_1 + W_2), \\ \sigma_{33} &= -p + 2W_1 + 2(2+\gamma^2)W_2, \quad \sigma_{13} = \sigma_{23} = 0, \end{aligned} \tag{3.10}$$

evaluated for $I_1 = I_2 = 3 + \gamma^2$.

In the (X_1, X_2)-plane the Eulerian principal axes $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ are given by

$$\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2, \quad \mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2, \tag{3.11}$$

where $\mathbf{e}_1, \mathbf{e}_2$ are the Cartesian axes and the angle ϕ is given by

$$\tan 2\phi = 2/\gamma. \tag{3.12}$$

Since, for an isotropic material, $\boldsymbol{\sigma}$ is coaxial with the Eulerian principal axes its components may, alternatively, be given in terms of its principal values by

$$\sigma_{11} = \sigma_1 \cos^2\phi + \sigma_2 \sin^2\phi, \quad \sigma_{22} = \sigma_1 \sin^2\phi + \sigma_2 \cos^2\phi, \tag{3.13}$$

$$\sigma_{12} = (\sigma_1 - \sigma_2)\sin\phi\cos\phi, \quad \sigma_{33} = \sigma_3. \tag{3.14}$$

The principal Cauchy stresses are given in terms of the principal stretches, which, for the considered deformation, are $\lambda_1, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, where λ_1 is related to the amount of shear γ by

$$\lambda_1 - \lambda_1^{-1} = \gamma \tag{3.15}$$

and we have taken $\lambda_1 \geq 1$ to correspond to $\gamma \geq 0$.

Instead of regarding W as a function of I_1 and I_2 or of the stretches we may take it to be a function of γ and define

$$\bar{W}(\gamma) = \hat{W}(\lambda_1, \lambda_1^{-1}), \tag{3.16}$$

subject to (3.15). Then, we have simply

$$\sigma_{12} = \bar{W}'(\gamma), \quad \sigma_{11} - \sigma_{22} = \gamma\sigma_{12}. \tag{3.17}$$

The second equation in (3.17) is an example of a *universal relation*, i.e. a connection between the components of stress which holds irrespective of the specific form of strain-energy function considered (within, in this case, the class of incompressible isotropic elastic solids). A detailed treatment of universal relations, including discussion of Ericksen's celebrated results (Ericksen, 1954, 1955), is contained in Chapter 3 in this volume.

Simple shear is an important example of a finite deformation. A general discussion of *shear* in the finite deformation context is provided in Chapter 6.

1.3.1.2 Fibre-reinforced materials

Again we consider the pure homogeneous strain defined by (3.1) and now we include two fibre directions, symmetrically disposed in the (X_1, X_2)-plane and given by

$$\mathbf{M} = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2, \quad \mathbf{M}' = \cos\alpha\,\mathbf{e}_1 - \sin\alpha\,\mathbf{e}_2, \tag{3.18}$$

where the angle α is constant and $\mathbf{e}_1, \mathbf{e}_2$ denote the Cartesian coordinate directions. The invariants I_1, I_2 are given by (2.68) and (2.69) subject to the incompressibility constraint (3.3), while with the definitions in (2.111) and (2.114) we use (3.18) to obtain

$$I_4 = I_6 = \lambda_1^2\cos^2\alpha + \lambda_2^2\sin^2\alpha, \quad I_5 = I_7 = \lambda_1^4\cos^2\alpha + \lambda_2^4\sin^2\alpha,$$

$$I_8 = \lambda_1^2\cos^2\alpha - \lambda_2^2\sin^2\alpha. \tag{3.19}$$

From (2.115) we then calculate the components of $\boldsymbol{\sigma}$ as

$$\sigma_{11} = -p + 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^4) + 2(W_4 + W_6 + W_8)\lambda_1^2\cos^2\alpha + 4(W_5 + W_7)\lambda_1^4\cos^2\alpha, \tag{3.20}$$

$$\sigma_{22} = -p + 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_2^4) + 2(W_4 + W_6 - W_8)\lambda_2^2\sin^2\alpha + 4(W_5 + W_7)\lambda_2^4\sin^2\alpha, \tag{3.21}$$

$$\sigma_{12} = 2[W_4 - W_6 + (W_5 - W_7)(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\alpha\cos\alpha, \tag{3.22}$$

$$\sigma_{33} = -p + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4), \quad \sigma_{13} = \sigma_{23} = 0. \tag{3.23}$$

In general, since $\sigma_{12} \neq 0$, shear stresses are required to maintain the pure homogeneous deformation and the principal axes of stress do not coincide with the Cartesian axes. However, in the special case in which the two families of fibres are mechanically equivalent the strain energy must be symmetric with respect to interchange of I_4 and I_6 and of I_5 and I_7. Since, for the considered deformation, we have $I_4 = I_6, I_5 = I_7$ it follows that $W_4 = W_6, W_5 = W_7$ and hence that $\sigma_{12} = 0$. The principal axes of stress then coincide with the Cartesian axes and $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are the principal Cauchy stresses.

Since I_1, I_2 depend symmetrically on $\lambda_1, \lambda_2, \lambda_3$ and I_4, I_5, I_8 depend on λ_1, λ_2 and α, we may regard the strain energy as a function of $\lambda_1, \lambda_2, \lambda_3$, subject to (3.3), and α. We write $W(\lambda_1, \lambda_2, \lambda_3, \alpha)$, but it should be emphasized that, unlike in the isotropic case, W is *not* symmetric with respect to interchange of any pair of the stretches. It is straightforward to check that

$$\sigma_{11} \equiv \sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p, \quad \sigma_{22} \equiv \sigma_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p, \quad \sigma_{33} \equiv \sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - p. \tag{3.24}$$

As in the isotropic case we make use of (3.3) to recast the strain energy as a function of λ_1 and λ_2 and define

$$\hat{W}(\lambda_1, \lambda_2, \alpha) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}, \alpha), \tag{3.25}$$

which is not symmetric in λ_1, λ_2 in general. Then, we obtain from (3.24)

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{3.26}$$

which are identical *in form* (but not in content) to equations (3.5).

For the simple shear deformation (3.9) the situation is more complicated. The invariants are given by

$$\begin{aligned}
&I_1 = I_2 = 3 + \gamma^2, \quad I_3 = 1, \\
&I_4 = 1 + \gamma \sin 2\alpha + \gamma^2 \sin^2 \alpha, \quad I_6 = 1 - \gamma \sin 2\alpha + \gamma^2 \sin^2 \alpha, \\
&I_5 = (1 + \gamma^2) \cos^2 \alpha + 2\gamma(2 + \gamma^2) \sin \alpha \cos \alpha + (\gamma^4 + 3\gamma^2 + 1) \sin^2 \alpha, \\
&I_7 = (1 + \gamma^2) \cos^2 \alpha - 2\gamma(2 + \gamma^2) \sin \alpha \cos \alpha + (\gamma^4 + 3\gamma^2 + 1) \sin^2 \alpha, \\
&I_8 = \cos^2 \alpha - (1 + \gamma^2) \sin^2 \alpha.
\end{aligned} \tag{3.27}$$

The components of the Cauchy stress tensor are calculated as

$$\begin{aligned}
\sigma_{11} = {} & -p + 2W_1(1 + \gamma^2) + 2W_2(2 + \gamma^2) \\
& + 2[W_4 + W_6 + W_8 + 2(W_5 + W_7)(1 + \gamma^2)] \cos^2 \alpha \\
& + 4[W_4 - W_6 + (W_5 - W_7)(3 + \gamma^2)]\gamma \sin \alpha \cos \alpha \\
& + 2[W_4 + W_6 - W_8 + 2(W_5 + W_7)(2 + \gamma^2)]\gamma^2 \sin^2 \alpha, \\
\sigma_{22} = {} & -p + 4W_1 + 4W_2 + 2(W_4 + W_6 - W_8) \sin^2 \alpha \\
& + 4(W_5 - W_7)\gamma \sin \alpha \cos \alpha + 4(W_5 + W_7)(1 + \gamma^2) \sin^2 \alpha, \\
\sigma_{12} = {} & 2(W_1 + W_2)\gamma + 2(W_4 - W_6) \sin \alpha \cos \alpha + 2(W_4 + W_6 \\
& - W_8)\gamma \sin^2 \alpha + 2(W_5 + W_7)\gamma[\cos^2 \alpha + (3 + \gamma^2) \sin^2 \alpha], \\
\sigma_{33} = {} & -p + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4), \quad \sigma_{13} = \sigma_{23} = 0.
\end{aligned} \tag{3.28}$$

By defining $\bar{W}(\gamma, \alpha)$ analogously to (3.18) through the invariants (3.27) with (3.17), it is straightforward to show that

$$\sigma_{12} = \frac{\partial \bar{W}}{\partial \gamma}, \tag{3.29}$$

exactly as in the isotropic case. Here, however, the universal relation $(3.17)_2$ does not hold. This is a reflection of the fact that the principal axes of Cauchy stress do not coincide with the Eulerian axes. Indeed, if ϕ^* denotes the analogue of ϕ, which is given by (3.12), for the principal stress axes then, in general, ϕ^* is given by

$$\tan 2\phi^* = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}, \tag{3.30}$$

and, since the right-hand side of (3.30) is not equal to $2/\gamma$, we see that $\phi^* \neq \phi$.

1.3.2 Extension and inflation of a thick-walled tube

In this section we examine an example of a *non-homogeneous* deformation. Other examples can be found in the texts cited in Section 1.1. We consider a thick-walled circular cylindrical tube whose initial geometry is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{3.31}$$

where A, B, L are positive constants and R, Θ, Z are cylindrical polar coordinates. The tube is deformed so that the circular cylindrical shape is maintained, and the material of the tube is taken to be incompressible. The resulting deformation is then described by the equations

$$r^2 - a^2 = \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{3.32}$$

where r, θ, z are cylindrical polar coordinates in the deformed configuration, λ_z is the axial stretch and a is the internal radius of the deformed tube.

The principal stretches $\lambda_1, \lambda_2, \lambda_3$ are associated respectively with the radial, azimuthal and axial directions and are written

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} = \lambda, \quad \lambda_3 = \lambda_z, \tag{3.33}$$

wherein the notation λ is introduced. It follows from (3.32) and (3.33) that

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{3.34}$$

where

$$\lambda_a = a/A, \quad \lambda_b = b/B. \tag{3.35}$$

For a fixed value of λ_z the inequalities

$$\lambda_a^2\lambda_z \geq 1, \quad \lambda_a \geq \lambda \geq \lambda_b \tag{3.36}$$

hold during inflation of the tube, with equality holding if and only if $\lambda = \lambda_z^{-1/2}$ for $A \leq R \leq B$.

We use the notation (3.4) for the strain energy but with $\lambda_2 = \lambda$ and $\lambda_3 = \lambda_z$ as the independent stretches, so that

$$\hat{W}(\lambda, \lambda_z) = W(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z). \tag{3.37}$$

Hence

$$\sigma_2 - \sigma_1 = \lambda \hat{W}_\lambda, \quad \sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z}, \tag{3.38}$$

where the subscripts indicate partial derivatives.

The equilibrium equation (2.30) in the absence of body forces reduces to the single scalar equation

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0 \tag{3.39}$$

in terms of the principal Cauchy stresses, and to this are adjoined the boundary conditions

$$\sigma_1 = \begin{cases} -P & \text{on } R = A \\ 0 & \text{on } R = B \end{cases} \tag{3.40}$$

corresponding to pressure $P\,(\geq 0)$ on the inside of the tube and zero traction on the outside.

By making use of (3.32)–(3.35) the independent variable may be changed from r to λ and integration of (3.39) and application of the boundary conditions (3.40) yields

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda} \mathrm{d}\lambda. \tag{3.41}$$

Since, from (3.34), λ_b depends on λ_a, equation (3.41) provides an expression for P as a function of λ_a when λ_z is fixed. In order to hold λ_z fixed an axial load, N say, must be applied to the ends of the tube. This is given by

$$N/\pi A^2 = (\lambda_a^2\lambda_z - 1) \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2} \Big(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda \frac{\partial \hat{W}}{\partial \lambda}\Big) \lambda \mathrm{d}\lambda + P\lambda_a^2. \tag{3.42}$$

For a thin-walled (membrane) tube the above results may be simplified since integration through the wall thickness is no longer needed. A general account of nonlinear elastic *membrane theory* is contained in Chapter 7, while Chapter 8 is concerned with *elastic surfaces.*

By analogy with the analysis in Section 1.3.1.2, for fibre reinforced materials with the fibre directions $\mathbf{M}$ and $\mathbf{M}'$ locally in the (Θ, Z)-plane symmetrically disposed with respect to the axial direction the strain energy may be written in the form

$$\hat{W}(\lambda, \lambda_z, \alpha), \tag{3.43}$$

where, we recall, $\hat{W}$ is not symmetric in λ and λ_z. Furthermore the formulas (3.41) and (3.42) again apply and are valid if the fibre directions depend on the radius, i.e. if α depends on R.

1.4 Incremental equations

1.4.1 Incremental deformation

Suppose that a solution χ to the boundary-value problem (2.116)–(2.118) is known and consider the problem of finding solutions near to χ when the boundary conditions are perturbed. Let χ' be a solution for the perturbed problem and write $\mathbf{x}' = \chi'(\mathbf{X})$. Also, we write

$$\dot{\mathbf{x}} = \mathbf{x}' - \mathbf{x} = \chi'(\mathbf{X}) - \chi(\mathbf{X}) \equiv \dot{\chi}(\mathbf{X}) \tag{4.1}$$

for the difference of the solutions, so that

$$\operatorname{Grad}\dot{\chi} = \operatorname{Grad}\chi' - \operatorname{Grad}\chi \equiv \dot{\mathbf{F}}. \tag{4.2}$$

Note that $\dot{\mathbf{F}}$ is linear in $\dot{\chi}$ and that in this expression no approximation has been made. In what follows, however, we shall consider approximations in which all terms are linearized in the incremental deformation $\dot{\mathbf{x}}$ and its gradient (4.2). Quantities with a superposed dot indicate the appropriate linearization.

From (2.17) we calculate the increment $\dot{\mathbf{F}}$ in terms of the increments of $\mathbf{R}$ and $\mathbf{U}$ in the form

$$\mathbf{R}^T\dot{\mathbf{F}} = \mathbf{R}^T\dot{\mathbf{R}}\mathbf{U} + \dot{\mathbf{U}}. \tag{4.3}$$

Next, we introduce the notations

$$\mathbf{\Omega}^R = \mathbf{R}^T\dot{\mathbf{R}}, \quad \mathbf{\Omega}^L = \sum_{i=1}^{3} \dot{\mathbf{u}}^{(i)} \otimes \mathbf{u}^{(i)} \tag{4.4}$$

for the incremental rotations associated with $\mathbf{R}$ and with the Lagrangian principal axes respectively. Then, from (2.18), we calculate

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} + \mathbf{\Omega}^L\mathbf{U} - \mathbf{U}\mathbf{\Omega}^L, \tag{4.5}$$

and hence the components of $\dot{\mathbf{U}}$ and $\mathbf{\Omega}^R$ on the Lagrangian principal axes are obtained in the form

$$(\dot{\mathbf{U}})_{ii} = \dot{\lambda}_i, \quad (\dot{\mathbf{U}})_{ij} = \Omega^L_{ij}(\lambda_j - \lambda_i) \quad i \neq j, \tag{4.6}$$

$$(\mathbf{\Omega}^R\mathbf{U})_{ij} = \Omega^R_{ij}\lambda_j, \quad (\mathbf{\Omega}^R\mathbf{U})_{ji} = -\Omega^R_{ij}\lambda_i \quad i \neq j. \tag{4.7}$$

1.4.2 Incremental stress and equilibrium

The nominal stress difference is

$$\dot{\mathbf{S}} = \mathbf{S}' - \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}') - \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}), \tag{4.8}$$

which has the linear approximation

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}^1 \dot{\mathbf{F}}, \tag{4.9}$$

where

$$\boldsymbol{\mathcal{A}}^1 = \frac{\partial^2 W}{\partial \mathbf{F}^2} \tag{4.10}$$

is the (fourth-order) tensor of *elastic moduli* associated with the pair $(\mathbf{S}, \mathbf{F})$ of stress and deformation tensors. In components, (4.10) is given by (2.121). The component form of (4.9) is

$$\dot{S}_{\alpha i} = \mathcal{A}^1_{\alpha i \beta j} \dot{F}_{j\beta}, \tag{4.11}$$

and this serves to define the product appearing in (4.9).

For an incompressible material it follows by taking the increment of the first equation in (2.85) that the counterpart of (4.9) is

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}^1 \dot{\mathbf{F}} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{4.12}$$

and this is coupled with the (linearized) incremental form

$$\operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0 \tag{4.13}$$

of the incompressibility condition $\det \mathbf{F} = 1$, where $\dot{p}$ is the (linearized) incremental form of p. For an incompressible material the definition (4.10) remains valid subject to the constraint $\det \mathbf{F} = 1$.

From the equilibrium equation (2.31) and its counterpart for χ', we obtain, by subtraction,

$$\operatorname{Div} \dot{\mathbf{S}} + \rho_r \dot{\mathbf{b}} = \mathbf{0}. \tag{4.14}$$

This is *exact*, but in the linear approximation $\dot{\mathbf{S}}$ is replaced by either (4.9), or (4.12) with (4.13), as appropriate and with $\dot{\mathbf{b}}$ linearized in $\dot{\chi}$.

Let $\dot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\tau}}$ be the prescribed data for the incremental deformation $\dot{\chi}$. Then, the incremental versions of the boundary conditions (2.117) and (2.118) are written

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\xi}} \quad \text{on } \partial\mathcal{B}_r^x, \qquad \dot{\mathbf{S}}^T\mathbf{N} = \dot{\boldsymbol{\tau}} \quad \text{on } \partial\mathcal{B}_r^\tau. \tag{4.15}$$

Together, the equations (4.14), with (4.9) for an unconstrained material, or (4.12) with (4.13) for an incompressible material, (4.2) and the boundary conditions (4.15) constitute the basic boundary-value problem of incremental elasticity given that the underlying deformation χ is known. The equations are also referred to as the equations of small deformations superimposed on a finite (or large) deformation. The linearized equations constitute the first-order terms associated with a formal perturbation expansion in the incremental deformation. The higher-order (nonlinear) terms are required for weakly nonlinear analysis of the stability of finitely deformed configurations and this topic is addressed in Chapter 10 in this volume; related work is discussed in the papers by Fu and Rogerson (1994), Fu (1995, 1998), Ogden and Fu (1996) and Fu and Ogden (1999), for example. For discussion of the mathematical structure of the incremental equations see, for example, Hayes and Horgan (1974).

In Section 1.5 the incremental equations (4.14) and boundary conditions (4.15) will be specialized for plane strain deformations and for an incompressible material in order to discuss an illustrative prototype example of a problem involving incremental deformations and bifurcation from a finitely deformed configuration.

1.4.2.1 Elastic moduli for an isotropic material

For the important special case of an isotropic material it is useful to give explicit expressions for the components of $\boldsymbol{\mathcal{A}}^1$. These are obtained by referring equation (4.9) to principal axes and making use of equations (4.3)–(4.7). Thus, for an isotropic elastic material, the (non-zero) components of $\boldsymbol{\mathcal{A}}^1$ referred to the principal axes $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are given by

$$\mathcal{A}^1_{iijj} = W_{ij}, \tag{4.16}$$

$$\mathcal{A}^1_{ijij} - \mathcal{A}^1_{ijji} = \frac{W_i + W_j}{\lambda_i + \lambda_j} \quad i \neq j, \tag{4.17}$$

$$\mathcal{A}^1_{ijij} + \mathcal{A}^1_{ijji} = \frac{W_i - W_j}{\lambda_i - \lambda_j} \quad i \neq j,\ \lambda_i \neq \lambda_j, \tag{4.18}$$

$$\mathcal{A}^1_{ijij} + \mathcal{A}^1_{ijji} = W_{ii} - W_{ij} \quad i \neq j,\ \lambda_i = \lambda_j, \tag{4.19}$$

where $W_i = \partial W/\partial\lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$, $i, j \in \{1,2,3\}$, and no summation is implied by the repetition of indices. In (4.16)–(4.19) the convention of using Greek letters for indices relating to Lagrangian components has been dropped. For details of the derivation of these components we refer to Ogden (1997). Equations (4.16)–(4.19) apply for both compressible and incompressible materials subject, in the latter case, to the constraint (3.3). Expressions analogous to (4.16)–(4.19) for the components of the tensors of moduli associated with conjugate variables based on the class of strain tensors (2.23)–(2.24) are obtainable in a similar way, but details are not given here. The appropriate calculations are illustrated in, for example, Chadwick and Ogden (1971) and Ogden (1974a, b, 1997) for first- and second-order moduli, while in Fu and Ogden (1999) the corresponding calculations for the first-, second- and third-order moduli are provided relative to the current configuration. For fibre-reinforced materials an expression for the elastic modulus tensor associated with the Green strain tensor $\mathbf{E}$ defined in (2.16) and its conjugate (second Piola-Kirchhoff) stress $\mathbf{T}^{(2)}$ defined in (2.36) is given by Holzapfel and Gasser (2000), in which the theory is extended to consideration of viscoelasticity.

In the classical theory of elasticity, corresponding to the situation in which there is no underlying deformation or stress, the components of $\boldsymbol{\mathcal{A}}^1$ can be written compactly in the form

$$\mathcal{A}^1_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.20}$$

where λ and μ are the classical Lamé moduli of elasticity and δ_{ij} is the Kronecker delta. Note that λ is not used subsequently in this chapter so no confusion with the notation used for the stretches should arise. The values of W_{ij} when $\lambda_i = 1$ for $i \in \{1,2,3\}$ are simply $W_{ii} = \lambda + 2\mu$, $W_{ij} = \lambda$, $i \neq j$. Also, we take $W_i = 0$ when $\lambda_j = 1$ for $i, j \in \{1,2,3\}$ so that the configuration $\mathcal{B}_r$ is stress free (a *natural configuration*).

The counterpart of (4.20) for an incompressible material is

$$\mathcal{A}^1_{iiii} = \mathcal{A}^1_{ijij} = \mu, \quad \mathcal{A}^1_{iijj} = \mathcal{A}^1_{ijji} = 0 \quad i \neq j, \tag{4.21}$$

and $W_{ii} = W_i = \mu$, $W_{ij} = 0$, where μ is the *shear modulus* in $\mathcal{B}_r$. These expressions are not unique because they depend on the point at which $\lambda_1\lambda_2\lambda_3$ is set to unity in the differentiation. The differences between (4.21) and any alternative expressions are accounted for by the incremental Lagrange multiplier $\dot{p}$ in (4.12). In terms of the strain-energy function $\hat{W}$ defined in (3.4) the restrictions required in $\mathcal{B}_r$ may be written

$$\hat{W}(1,1) = 0, \quad \hat{W}_\alpha(1,1) = 0, \quad \hat{W}_{12}(1,1) = 2\mu, \quad \hat{W}_{\alpha\alpha}(1,1) = 4\mu, \tag{4.22}$$

where the index α is 1 or 2.

1.4.2.2 Strong ellipticity for an isotropic material

We now give an explicit form for the strong ellipticity inequality in two dimensions for both compressible and incompressible materials. For this purpose we refer $\boldsymbol{\mathcal{A}}^1$ to principal axes and make use of the expressions (4.16)–(4.19) and the formulas (4.3)–(4.7). By restricting attention to the (1,2) principal plane it can be seen that necessary and sufficient conditions for the resulting quadratic form in the incremental quantities to be strictly positive are jointly

$$W_{11} > 0, \quad W_{22} > 0, \quad \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1 - \lambda_2} > 0, \tag{4.23}$$

$$(W_{11}W_{22})^{1/2} - W_{12} + \frac{W_1 + W_2}{\lambda_1 + \lambda_2} > 0, \tag{4.24}$$

$$(W_{11}W_{22})^{1/2} + W_{12} + \frac{W_1 - W_2}{\lambda_1 - \lambda_2} > 0. \tag{4.25}$$

See, for example, Knowles and Sternberg (1977), Hill (1979), Dowaikh and Ogden (1991), Davies (1991) and Wang and Aron (1996).

For an incompressible material we then use (2.125) in two dimensions to write $m_1 = n_2, m_2 = -n_1$, so that (2.127) reduces to

$$\mathcal{A}^1_{01212}n_1^4 + (\mathcal{A}^1_{01111} + \mathcal{A}^1_{02222} - 2\mathcal{A}^1_{01122} - 2\mathcal{A}^1_{02112})n_1^2n_2^2 + \mathcal{A}^1_{02121}n_2^4 > 0, \tag{4.26}$$

and on use of (4.16)–(4.18) and the specialization of (2.128) to principal axes with $J = 1$, we obtain

$$\mathcal{A}^1_{01111} = \lambda_1^2 W_{11}, \quad \mathcal{A}^1_{01122} = \lambda_1\lambda_2 W_{12}, \quad \mathcal{A}^1_{02222} = \lambda_2^2 W_{22}, \tag{4.27}$$

$$\lambda_1^{-2}\mathcal{A}^1_{01212} = \lambda_2^{-2}\mathcal{A}^1_{02121} = \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1^2 - \lambda_2^2}, \quad \mathcal{A}^1_{02112} = \frac{\lambda_2 W_1 - \lambda_1 W_2}{\lambda_1^2 - \lambda_2^2}\lambda_1\lambda_2. \tag{4.28}$$

Necessary and sufficient conditions for (4.26) to hold are then easily seen to be

$$\mathcal{A}^1_{01212} > 0, \quad \mathcal{A}^1_{01111} + \mathcal{A}^1_{02222} - 2\mathcal{A}^1_{01122} - 2\mathcal{A}^1_{02112} > -2\sqrt{\mathcal{A}^1_{01212}\mathcal{A}^1_{02121}}, \tag{4.29}$$

and we note that these are independent of p.

In terms of the strain-energy function the inequalities (4.29) may be written

$$\frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1 - \lambda_2} > 0, \quad \lambda_1^2 W_{11} - 2\lambda_1\lambda_2 W_{12} + \lambda_2^2 W_{22} + 2\lambda_1\lambda_2\frac{W_1 + W_2}{\lambda_1 + \lambda_2} > 0. \tag{4.30}$$

1.4.3 Incremental uniqueness and stability

We next examine the question of uniqueness of solution of the incremental problem and the associated question of stability of the deformation χ. We focus first on the theoretical development for unconstrained materials, and for simplicity we restrict attention to the dead-load boundary-value problem with no body forces. We also take the boundary conditions to be homogeneous, so that $\dot{\boldsymbol{\xi}} = \mathbf{0}, \dot{\boldsymbol{\tau}} = \mathbf{0}$ in (4.15). The appropriate specialization of (4.14) and (4.15) is then

$$\operatorname{Div}\dot{\mathbf{S}} = \mathbf{0} \quad \text{in } \mathcal{B}_r, \quad \dot{\boldsymbol{\chi}} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_r^x, \quad \dot{\mathbf{S}}^T\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_r^\tau. \tag{4.31}$$

One solution of this is $\dot{\boldsymbol{\chi}} = \mathbf{0}$. We therefore wish to determine whether this is the only solution. With this aim in mind we consider next the change in the energy functional $E\{\boldsymbol{\chi}\}$, defined by (2.133), due to the change in the deformation from $\boldsymbol{\chi}$ to $\boldsymbol{\chi}'$, the body-force term being omitted. On use of the boundary condition $(4.31)_2$ and the divergence theorem this is seen to be

$$E\{\boldsymbol{\chi}'\} - E\{\boldsymbol{\chi}\} = \int_{\mathcal{B}_r} \{W(\mathbf{F}') - W(\mathbf{F}) - \operatorname{tr}(\mathbf{S}\dot{\mathbf{F}})\}\,\mathrm{d}V. \tag{4.32}$$

By application of the Taylor expansion to $W(\mathbf{F}')$ this then becomes

$$E\{\boldsymbol{\chi}'\} - E\{\boldsymbol{\chi}\} = \frac{1}{2}\int_{\mathcal{B}_r} \operatorname{tr}(\dot{\mathbf{S}}\dot{\mathbf{F}})\,\mathrm{d}V \equiv \frac{1}{2}\int_{\mathcal{B}_r} \operatorname{tr}\{(\boldsymbol{\mathcal{A}}^1\dot{\mathbf{F}})\dot{\mathbf{F}}\}\,\mathrm{d}V, \tag{4.33}$$

correct to the second order in $\dot{\mathbf{F}}$.

If the inequality

$$\int_{\mathcal{B}_r} \operatorname{tr}(\dot{\mathbf{S}}\dot{\mathbf{F}})\,\mathrm{d}V \equiv \int_{\mathcal{B}_r} \operatorname{tr}\{(\boldsymbol{\mathcal{A}}^1\dot{\mathbf{F}})\dot{\mathbf{F}}\}\,\mathrm{d}V > 0 \tag{4.34}$$

holds for all $\dot{\boldsymbol{\chi}} \not\equiv \mathbf{0}$ in $\mathcal{B}_r$ satisfying $\dot{\boldsymbol{\chi}} = \mathbf{0}$ on $\partial\mathcal{B}_r^x$ then, to the second order in $\dot{\mathbf{F}}$, (4.33) implies that $E\{\boldsymbol{\chi}'\} > E\{\boldsymbol{\chi}\}$ for all admissible $\dot{\boldsymbol{\chi}} \not\equiv \mathbf{0}$ satisfying $(4.31)_2$. This inequality states that $\boldsymbol{\chi}$ is *locally stable* with respect to perturbations $\dot{\boldsymbol{\chi}}$ from $\boldsymbol{\chi}$, and that $\boldsymbol{\chi}$ is a *local minimizer* of the energy functional. Furthermore, if (4.34) holds in the configuration $\mathcal{B}$ then the only solution of the homogeneous incremental problem is the trivial solution $\dot{\boldsymbol{\chi}} \equiv \mathbf{0}$, i.e. non-trivial solutions are excluded.

This can be seen by noting that if $\dot{\boldsymbol{\chi}} \not\equiv \mathbf{0}$ is a non-trivial solution then, by the divergence theorem and use of equations (4.31), it follows that

$$\int_{\mathcal{B}_r} \operatorname{tr}(\dot{\mathbf{S}}\dot{\mathbf{F}})\,\mathrm{d}V = 0 \tag{4.35}$$

necessarily holds. The inequality (4.34) is referred to as the *exclusion condition* (for the dead-load traction boundary condition). The trivial solution is then the unique solution. Thus, on a path of deformation corresponding to mixed dead-load and placement boundary conditions bifurcation of solutions is excluded provided (4.34) holds. This exclusion condition requires modification if $\boldsymbol{\tau}$ is allowed to depend on the deformation. Generally, the exclusion condition will involve both a surface integral and a volume integral. For a discussion of connections between incremental (infinitesimal) stability and uniqueness we refer to Hill (1957), Truesdell and Noll (1965) and Beatty (1987).

For the all-round dead-load problem, $\partial\mathcal{B}_r^x = \emptyset, \partial\mathcal{B}_r = \partial\mathcal{B}_r^\tau$. If the underlying deformation is homogeneous then $\mathbf{F}$ is independent of $\mathbf{X}$ and so, therefore, is $\boldsymbol{\mathcal{A}}^1$. It then follows that (4.34) is equivalent to the local condition

$$\mathrm{tr}\,(\dot{\mathbf{S}}\dot{\mathbf{F}}) \equiv \mathrm{tr}\,\{(\boldsymbol{\mathcal{A}}^1\dot{\mathbf{F}})\dot{\mathbf{F}}\} > 0 \tag{4.36}$$

for all $\dot{\mathbf{F}} \neq \mathbf{0}$, i.e. $\boldsymbol{\mathcal{A}}^1$ is positive definite at each point $\mathbf{X} \in \mathcal{B}_r$. It is well known that the inequality (4.36) *cannot* hold in all configurations, and hence $\boldsymbol{\mathcal{A}}^1$ is singular in certain configurations when regarded as a linear mapping on the (nine-dimensional) space of increments $\dot{\mathbf{F}}$. In other words, in the incremental stress-deformation relation $\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}^1\dot{\mathbf{F}}$ the incremental stress $\dot{\mathbf{S}}$ can vanish for non-zero $\dot{\mathbf{F}}$ and bifurcation of the deformation path can occur. The reader is referred to, for example, Ogden (1991, 1997, 2000) for detailed analysis of the singularities of $\boldsymbol{\mathcal{A}}^1$ and their implications for bifurcation in the dead-load problem.

1.4.3.1 Specialization to isotropy

In the case of an isotropic material the local stability inequality (4.36) can be given an explicit form in terms of the derivatives of the strain-energy function with respect to the stretches. Use of the expressions (4.3)–(4.7) and (4.16)–(4.18) in (4.36) referred to principal axes leads to

$$\begin{aligned}\mathrm{tr}\,\{(\boldsymbol{\mathcal{A}}^1\dot{\mathbf{F}})\dot{\mathbf{F}}\} = \sum_{i,j=1}^{3} W_{ij}\dot{\lambda}_i\dot{\lambda}_j + \sum_{i\neq j}(W_i - W_j)(\lambda_i - \lambda_j)(\Omega_{ij}^L + \frac{1}{2}\Omega_{ij}^R)^2 \\ + \frac{1}{4}\sum_{i\neq j}(W_i + W_j)(\lambda_i + \lambda_j)(\Omega_{ij}^R)^2.\end{aligned} \tag{4.37}$$

Since $\dot{\lambda}_i$, Ω_{ij}^L, Ω_{ij}^R are independent, necessary and sufficient conditions for (4.36) are therefore

$$\text{matrix } (W_{ij}) \text{ is positive definite,} \tag{4.38}$$

$$W_i + W_j > 0 \qquad i \neq j, \tag{4.39}$$

$$\frac{W_i - W_j}{\lambda_i - \lambda_j} > 0 \qquad i \neq j \tag{4.40}$$

jointly for $i, j \in \{1, 2, 3\}$. Note that when $\lambda_i = \lambda_j$, $i \neq j$ (4.40) reduces to $W_{ii} - W_{ij} > 0$ and that (4.38) and (4.40) hold in the natural configuration provided the usual inequalities $\mu > 0$, $3\lambda + 2\mu > 0$ satisfied by the Lamé moduli hold.

On use of (4.12) and (4.13) we may deduce that the analogue of the stability inequality (4.36) for an incompressible material is

$$\mathrm{tr}\,\{(\mathcal{A}^1\dot{\mathbf{F}})\dot{\mathbf{F}}\} + p\,\mathrm{tr}\,(\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\dot{\mathbf{F}}) > 0, \tag{4.41}$$

subject to (4.13). Note that this does not depend on $\dot{p}$.

In terms of the modified strain-energy function $\hat{W}(\lambda_1, \lambda_2)$ defined by (3.4), with p eliminated in favour of σ_3, the inequality (4.41) becomes explicitly

$$(\lambda_1^2\hat{W}_{11} - 2\sigma_3)\Big(\frac{\dot{\lambda}_1}{\lambda_1}\Big)^2 + (\lambda_2^2\hat{W}_{22} - 2\sigma_3)\Big(\frac{\dot{\lambda}_2}{\lambda_2}\Big)^2 + 2(\lambda_1\lambda_2\hat{W}_{12} - \sigma_3)\frac{\dot{\lambda}_1\dot{\lambda}_2}{\lambda_1\lambda_2}$$

$$+\sum_{i\neq j}(t_i - t_j)(\lambda_i - \lambda_j)(\Omega^L_{ij} + \frac{1}{2}\Omega^R_{ij})^2 + \frac{1}{4}\sum_{i\neq j}(t_i + t_j)(\lambda_i + \lambda_j)(\Omega^R_{ij})^2 > 0, \tag{4.42}$$

where the subscripts on $\hat{W}$ denote partial derivatives with respect to λ_1 and λ_2 and implicitly the connections (3.5) have been used.

The counterparts of (4.38)–(4.40) in this case are then

$$\text{matrix} \begin{bmatrix} \lambda_1^2\hat{W}_{11} - 2\sigma_3 & \lambda_1\lambda_2\hat{W}_{12} - \sigma_3 \\ \lambda_1\lambda_2\hat{W}_{12} - \sigma_3 & \lambda_2^2\hat{W}_{22} - 2\sigma_3 \end{bmatrix} \text{ is positive definite,} \tag{4.43}$$

$$t_i + t_j > 0 \quad i \neq j, \tag{4.44}$$

$$\frac{t_i - t_j}{\lambda_i - \lambda_j} > 0 \quad i \neq j. \tag{4.45}$$

Note that p occurs in (4.43)–(4.45) implicitly through σ_3 and t_i, $i \in \{1, 2, 3\}$.

In general the stability inequality (4.36) is stronger than the strong ellipticity inequality (2.124). This can be seen by making the specialization $\dot{\mathbf{F}} = \mathbf{m} \otimes \mathbf{N}$ in (4.36), which then (in component form) reduces to (2.124) . This is also the case for incompressible materials, for which use of (2.125) and (2.126) enables the term in p to be removed from (4.41).

1.4.4 Global non-uniqueness

The singularities of $\mathcal{A}^1$ mentioned above are local manifestations of the global non-uniqueness in the relationship between $\mathbf{S}$ and $\mathbf{F}$ expressed through the constitutive equation

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) \tag{4.46}$$

for an unconstrained material, or $(2.85)_1$ in the case of an incompressible material. For an isotropic material we recall from (2.101) the polar decomposition

$$\mathbf{S} = \mathbf{T}\mathbf{R}^T. \tag{4.47}$$

As mentioned in Section 1.2.4.2 this decomposition is not unique since $\mathbf{T}$ is not sign definite. We summarize briefly the extent of non-uniqueness and refer the reader to Ogden (1977, 1997, 2000) for detailed discussion. For a given $\mathbf{S}$ there are *four* distinct polar decompositions of the form (4.47) if $t_i^2 \neq t_j^2$, $i, j \in \{1,2,3\}$, and infinitely many when $t_i^2 = t_j^2$, $i \neq j$, where t_i are the principal Biot stresses. However, at most one of these satisfies the stability inequalities (4.39) or (4.44). For each such polar decomposition each $\mathbf{R}$ and $\mathbf{T}$ pair is determined uniquely if $t_i^2 \neq t_j^2$, $i, j \in \{1,2,3\}$ and to within an arbitrary rotation about the $\mathbf{u}^{(k)}$ principal axis if $t_i^2 = t_j^2$, $i \neq j$, where (i,j,k) is a permutation of $(1,2,3)$. Then, for each $\mathbf{T}$, $\mathbf{U}$ can in principle be found by inverting (2.88), or (2.89) for an incompressible material. Since, from (2.94), we have

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{4.48}$$

and $\mathbf{T}$ is coaxial with $\mathbf{U}$, this inversion is equivalent to inverting the scalar equations (2.95) or (2.96) for λ_i, $i \in \{1,2,3\}$, when t_i, $i \in \{1,2,3\}$, are given. The resulting deformation gradients are then calculated from $\mathbf{F} = \mathbf{R}\mathbf{U}$ since $\mathbf{u}^{(i)}$, $i \in \{1,2,3\}$, are determined from $\mathbf{T}$. In general, however, these inversions are not unique and the extent of their non-uniqueness is a separate question from that of non-uniqueness of the polar decomposition (4.47).

For an incompressible material, for example, elimination of p from (2.96) yields the equations

$$\lambda_1 t_1 - \lambda_1 \frac{\partial W}{\partial \lambda_1} = \lambda_2 t_2 - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \lambda_3 t_3 - \lambda_3 \frac{\partial W}{\partial \lambda_3}, \qquad \lambda_1\lambda_2\lambda_3 = 1, \tag{4.49}$$

from which $\lambda_1, \lambda_2, \lambda_3$ can, in principle, be determined when t_1, t_2, t_3 are prescribed. Examples illustrating non-uniqueness of the inversion of (4.49) are given in Ogden (1991, 1997, 2000).

The associated physical problem is that of a (dead-load) pure strain $\mathbf{F} = \mathbf{U}$ ($\mathbf{R} = \mathbf{I}$) in which the principal directions $\mathbf{u}^{(i)}$ are fixed as the load increases. The prototype of this problem is the Rivlin cube problem (Rivlin, 1974), for which $t_1 = t_2 = t_3$ and a cube of elastic material is subjected to equal normal forces per unit reference area on its three pairs of faces. Several variants of this problem and related problems are analyzed in Sawyers (1976), Ball and Schaeffer (1983), Ogden (1984c, 1985, 1987, 1997), MacSithigh (1986), Kearsley (1986) Chen (1987, 1988, 1995, 1996) and MacSithigh and Chen (1992a, b) amongst others. For a general discussion of bifurcation we refer to Chapter 9 in this volume.

1.5 Incremental boundary-value problems

Recalling the notation $\dot{\mathbf{x}} = \dot{\chi}(\mathbf{X})$ defined in (4.1) for the increment in $\mathbf{x}$ we now change the independent variable from $\mathbf{X}$ to $\mathbf{x}$ and introduce the *incremental displacement vector* $\mathbf{u}$ defined, through (2.2), as a function of $\mathbf{x}$ by

$$\mathbf{u}(\mathbf{x}) = \dot{\chi}(\chi^{-1}(\mathbf{x})). \tag{5.1}$$

The *displacement gradient* grad $\mathbf{u}$, which we denote by $\mathbf{\Gamma}$ is then given by

$$\mathbf{\Gamma} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{5.2}$$

so that it is the push forward (from $\mathcal{B}_r$ to $\mathcal{B}$) of the increment in $\mathbf{F}$. From (4.13) it follows that the incremental incompressibility condition may be written

$$\operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) \equiv \operatorname{tr}(\mathbf{\Gamma}) \equiv \operatorname{div}\mathbf{u} = 0. \tag{5.3}$$

The corresponding update (or push forward) from $\mathcal{B}_r$ to $\mathcal{B}$ of the incremental nominal stress, denoted $\mathbf{\Sigma}$, is defined by $\dot{\mathbf{S}}^T\mathbf{N}\mathrm{d}A = \mathbf{\Sigma}^T\mathbf{n}\mathrm{d}a$, where $\mathbf{N}\mathrm{d}A$ and $\mathbf{n}\mathrm{d}a$ are surface elements in $\mathcal{B}_r$ and $\mathcal{B}$, respectively. With the aid of (4.12) and Nanson's formula (2.8), we obtain

$$\mathbf{\Sigma} = J^{-1}\mathbf{F}\dot{\mathbf{S}} = \mathcal{A}_0^1\mathbf{\Gamma} + p\,\mathbf{\Gamma} - \dot{p}\,\mathbf{I}, \tag{5.4}$$

where $\mathcal{A}_0^1$ is the fourth-order (Eulerian) tensor whose components are given in terms of those of $\mathcal{A}^1$ by (2.128). With this updating the incremental equilibrium equation (4.14) becomes

$$\operatorname{div}\mathbf{\Sigma} = \mathbf{0}, \tag{5.5}$$

when body forces are omitted, where use has been made of $(2.11)_2$.

In component form equations (5.4) and (5.3) are combined to give

$$\Sigma_{ji} = \mathcal{A}_{0jilk}^1 u_{k,l} + p\,u_{j,i} - \dot{p}\,\delta_{ij}, \quad u_{i,i} = 0. \tag{5.6}$$

The incremental traction $\mathbf{\Sigma}^T\mathbf{n}$ per unit area on a surface in $\mathcal{B}$ with unit normal $\mathbf{n}$ has components

$$\Sigma_{ji}n_j = (\mathcal{A}_{0jilk}^1 + p\,\delta_{jk}\delta_{il})u_{k,l}n_j - \dot{p}\,n_i. \tag{5.7}$$

1.5.1 Plane incremental deformations

We now restrict attention to plane incremental deformations so that $u_3 = 0$ and u_1 and u_2 depend only on x_1 and x_2. Furthermore, we take the Cartesian axes to coincide with the Eulerian principal axes of the finite deformation associated with $\mathbf{F}$. With this restriction we deduce form $(5.6)_2$ the existence of a scalar function, ψ say, of x_1 and x_2 such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}. \tag{5.8}$$

Substitution of (5.8) into the equilibrium equation (5.5) with $(5.6)_1$ appropriately specialized then leads, after elimination of the terms in $\dot{p}$, to an equation for ψ. If the finite deformation is homogeneous and the material is isotropic, so that the components $\mathcal{A}^1_{0jilk}$ are constants and given by (4.27) and (4.28) then this equation has the compact form

$$\alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = 0, \tag{5.9}$$

where the coefficients are defined by

$$\alpha = \mathcal{A}^1_{01212}, \quad 2\beta = \mathcal{A}^1_{01111} + \mathcal{A}^1_{02222} - 2\mathcal{A}^1_{01122} - 2\mathcal{A}^1_{01221}, \quad \gamma = \mathcal{A}^1_{02121}. \tag{5.10}$$

For details of the derivation we refer to Dowaikh and Ogden (1990).

This is the incremental equilibrium equation for plane incremental deformations of an incompressible isotropic elastic solid in the (1, 2) principal plane for an arbitrary homogeneous deformation. For a specific incremental boundary-value problem appropriate boundary conditions need to be given. In order to illustrate these we now concentrate attention on the problem of incremental deformations of a homogeneously pure strained half-space.

1.5.1.1 Surface deformations of a half-space subject to pure homogeneous strain

We now consider the pure homogeneous strain

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \tag{5.11}$$

as in (3.1) and we take the deformed half-space $\mathcal{B}$ to be defined by $x_2 < 0$ with boundary $x_2 = 0$. On this boundary we set the boundary conditions to correspond to vanishing incremental traction. Specialization of (5.7) with $n_1 = 0, n_2 = 1, n_3 = 0$ then yields two equations. When expressed in terms of ψ after elimination of $\dot{p}$ by differentiation along the boundary and use of the equilibrium equation we obtain

$$\gamma(\psi_{,22} - \psi_{,11}) + \sigma_2\psi_{,11} = 0 \quad \text{on } x_2 = 0, \tag{5.12}$$

$$(2\beta + \gamma - \sigma_2)\psi_{,112} + \gamma\psi_{,222} = 0 \quad \text{on } x_2 = 0, \tag{5.13}$$

where σ_2 is the normal stress on the boundary (and uniform through $x_2 < 0$) associated with the underlying homogeneous deformation. Again we refer to Dowaikh and Ogden (1990) for details. Note that the coefficients (material constants) α, β, γ appearing in (5.9) also feature in the boundary conditions (5.12) and (5.13) but additionally σ_2 arises in the boundary conditions as a separate independent parameter.

In the notation (5.10) the strong ellipticity conditions (4.29) then take on the simple forms

$$\alpha > 0, \quad \beta > -\sqrt{\alpha\gamma}, \tag{5.14}$$

and these inequalities arise naturally in consideration of stability of the homogeneous deformation of the half-space, as we see below.

An incremental *surface deformation* must decay as $x_2 \to -\infty$. We consider ψ to have the form

$$\psi = A\exp(skx_2 - \mathrm{i}kx_1), \tag{5.15}$$

where A, s and k are constants. This is periodic in the x_1 direction. In general s is complex and is determined by substitution of (5.15) into (5.9), which yields the quadratic equation

$$\gamma s^4 - 2\beta s^2 + \alpha = 0 \tag{5.16}$$

for s^2. This equation has two solutions for s with positive real part, and we denote these by s_1, s_2. The general solution for ψ with the required decay properties is then

$$\psi = (A\mathrm{e}^{s_1kx_2} + B\mathrm{e}^{s_2kx_2})\mathrm{e}^{-\mathrm{i}kx_1}, \tag{5.17}$$

where A and B are constants.

After substitution of (5.17) into the boundary conditions (5.12) and (5.13) we deduce that there is a non-trivial solution for A and B (i.e. an incremental deformation is possible) if and only if the material constants α, β, γ and the normal stress σ_2 satisfy the equation

$$\alpha\gamma + 2\sqrt{\alpha\gamma}(\beta + \gamma - \sigma_2) - (\gamma - \sigma_2)^2 = 0. \tag{5.18}$$

Subject to the incompressibility condition (3.3) this identifies values of σ_2 and the stretches $\lambda_1, \lambda_2, \lambda_3$ for which bifurcation from the homogeneous deformation into an inhomogeneous mode of incremental deformation can occur.

If $\sigma_2 = 0$ in (5.18) it reduces to

$$\gamma[\alpha - \gamma + 2\sqrt{\alpha\gamma}(\beta + \gamma)] = 0. \tag{5.19}$$

Since, by (5.10) and (4.21), $\alpha = \gamma = \beta = \mu$ in the undeformed configuration both factors in (5.19) are positive there. Thus, incremental surfaces deformations are excluded in this configuration, and, by continuity, on any path of pure homogeneous strain from this configuration such that the inequalities

$$\gamma > 0, \quad \alpha - \gamma + 2\sqrt{\alpha\gamma}(\beta + \gamma) > 0 \tag{5.20}$$

hold. It is easy to show that the strong ellipticity inequalities (5.14) follow from (5.20). Note that (5.20) are weaker than the exclusion condition appropriate for all-round dead loading obtained by specializing (4.43)–(4.45) to the present two-dimensional situation since the boundary conditions considered here are different.

For $\sigma_2 \neq 0$ the exclusion condition is then seen to be

$$\alpha\gamma + 2\sqrt{\alpha\gamma}(\beta + \gamma - \sigma_2) - (\gamma - \sigma_2)^2 > 0, \tag{5.21}$$

which restricts σ_2 to a range of values dependent on the other parameters, which are subject to (5.20).

To illustrate the results graphically we restrict attention to plane strain with $\lambda_3 = 1$ and set $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}$ and define the strain energy in terms of λ through

$$\check{W}(\lambda) = W(\lambda, \lambda^{-1}, 1). \tag{5.22}$$

Then, (5.21) reduces to

$$\lambda^4 \check{W}''(\lambda) + \lambda \check{W}'(\lambda) - 2(\lambda^2 - 1)\sigma_2 - (\lambda^4 - 1)\sigma_2^2/\lambda\check{W}'(\lambda) > 0, \tag{5.23}$$

which puts restrictions on the allowable values of λ and σ_2. The curves in Figure 1 show the boundaries of the region defined by (5.23) in respect of the neo-Hookean strain-energy function (2.103) and a single-term strain-energy function in the class (2.106) given, in the present plane strain specialization, by

$$\check{W}(\lambda) = 8\mu(\lambda^{1/2} + \lambda^{-1/2} - 2). \tag{5.24}$$

In (a) the stable region is to the right of the left-hand curve and below the upper curve. In (b) the stable region is the area within the loop formed by the curve. In each case the natural configuration $\lambda = 1, \bar{\sigma}_2 = 0$ is within the stable region. If $\bar{\sigma}_2 = 0$ then the half-space is stable for $\lambda \geq 1$ but can become unstable in compression at a critical value of $\lambda < 1$ for the neo-Hookean material, while for the strain energy (5.24) stability may be lost in either compression or tension.

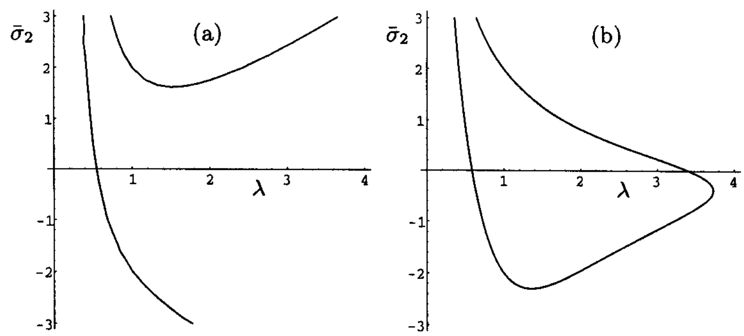

Fig. 1. Plot of the stable region in $(\lambda, \bar{\sigma}_2)$-space for (a) the neo-Hookean strain energy and (b) the strain energy (5.24).

The problem discussed here and some specializations have been examined by a number of authors, and we refer, in particular, to Nowinski (1969a, b), Usmani and Beatty (1974), Reddy (1982, 1983) and Dowaikh and Ogden (1990). For a compressible material the corresponding analysis is given by Dowaikh and Ogden (1991). See also Biot (1965). The analysis of stability of a thick plate subject to a pure homogeneous finite strain has been discussed by several authors and we mention, in particular, Sawyers and Rivlin (1974, 1982) and Sawyers (1977). Stability results are obtained in the static specialization of the problem of vibration of a finitely deformed plate by Ogden and Roxburgh (1993) and Roxburgh and Ogden (1994) for incompressible and unconstrained materials respectively, while the influence of a finite simple shear on stability has been examined on the same basis by Ogden and Connor (1995) and Connor and Ogden (1996).

References to the analysis of stability for problems involving the inflation of a thick-walled sphere or the extension and inflation of a thick-walled circular cylindrical tube can be found in Ogden (1997), for example. Recently, the stability of a rectangular block deformed into a sector of a circular cylindrical tube has been analyzed by Haughton (1999) and Dryburgh and Ogden (1999) from different points of view.

A comprehensive list of references to contributions on linear stability analysis is contained in an appendix at the end of Chapter 10 in this volume.

1.6 Elastodynamics

1.6.1 Kinematics

We now extend the analysis of the previous sections by allowing the deformation to depend on time. As before we take $\mathcal{B}_r$ to denote a fixed (time independent) reference configuration of the body (which may, but need not, be a configuration occupied by the body at some specific time). Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. With each $t \in I$ we associate a unique configuration $\mathcal{B}_t$ of the body. The (one-parameter) family of configurations $\{\mathcal{B}_t : t \in I\}$ is then called a *motion* of the body. We assume that as the body moves continuously then $\mathcal{B}_t$ changes continuously. As in Section 1.2, a point of $\mathcal{B}_r$ is labelled by its position vector $\mathbf{X}$. Let $\mathbf{x}$ be its position vector in the configuration $\mathcal{B}_t$ at time t, which is referred to as the *current configuration.*

Since $\mathcal{B}_t$ depends on t we write

$$\mathbf{x} = \chi_t(\mathbf{X}), \quad \mathbf{X} = \chi_t^{-1}(\mathbf{x}) \tag{6.1}$$

instead of (2.1) and (2.2), or

$$\mathbf{x} = \chi(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in \mathcal{B}_r,\ t \in I \tag{6.2}$$

in order to make the dependence on t explicit. It is usual to assume that $\chi(\mathbf{X}, t)$ is suitably regular, and for many purposes it may be taken to be twice continuously differentiable with respect to position and time.

The *velocity*, denoted $\mathbf{v}$, and *acceleration*, denoted $\mathbf{a}$, of a material point $\mathbf{X}$ are defined by

$$\mathbf{v} \equiv \mathbf{x}_{,t} = \frac{\partial}{\partial t}\chi(\mathbf{X}, t), \quad \mathbf{a} \equiv \mathbf{v}_{,t} \equiv \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\chi(\mathbf{X}, t), \tag{6.3}$$

respectively. We emphasize that $\partial/\partial t$ is the *material time derivative*, i.e. the time derivative at fixed $\mathbf{X}$, and it is denoted by $_{,t}$ when the independent variables are understood to be $\mathbf{X}$ and t.

Any scalar, vector or tensor field may be expressed in either the Eulerian description (as a function of $\mathbf{x}$ and t) or, equivalently, in the Lagrangian description (as a function of $\mathbf{X}$ and t) through the motion (6.2) or its inverse.

Thus, if the velocity $\mathbf{v}$ is regarded as a function of $\mathbf{x}$ and t, the *velocity gradient tensor*, denoted $\mathbf{L}$, is an Eulerian tensor defined as

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \quad L_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{6.4}$$

It follows that

$$\operatorname{Grad}\mathbf{x}_{,t} = \mathbf{F}_{,t} = (\operatorname{grad}\mathbf{v})\mathbf{F} = \mathbf{L}\mathbf{F}, \tag{6.5}$$

where the deformation gradient $\mathbf{F}$ is defined as in (2.3) but now depends on t. Equation (6.5) is analogous to the formula (5.2) in the context of incremental deformations.

Using the standard result

$$(\det \mathbf{F})_{,t} = (\det \mathbf{F})\mathrm{tr}\,(\mathbf{F}^{-1}\mathbf{F}_{,t}), \tag{6.6}$$

we deduce, using (6.5), that

$$J_{,t} \equiv (\det \mathbf{F})_{,t} = (\det \mathbf{F})\mathrm{tr}\,(\mathbf{L}) = J\mathrm{div}\,\mathbf{v}, \tag{6.7}$$

where $J = \det \mathbf{F}$, as in (2.5). Thus, $\mathrm{div}\,\mathbf{v}$ is a measure of the rate at which volume changes during the motion. For an *isochoric* motion $J \equiv 1$ and (6.7) reduces to

$$\mathrm{div}\,\mathbf{v} = 0. \tag{6.8}$$

Equation (6.8) is the analogue of the incompressibility condition (5.3) arising in the *linearized* incremental theory, but we note while (5.3) is a linear approximation equation (6.8) is exact in the dynamic context.

A rate counterpart of the mass conservation equation (2.25), which also applies when ρ and J depend on t, is obtained by differentiating (2.25) with respect to t and making use of (6.7) to give

$$\rho_{,t} + \rho\mathrm{div}\,\mathbf{v} = 0. \tag{6.9}$$

1.6.2 Equations of motion

The equation of motion in the form analogous to the equilibrium equation (2.31) is

$$\mathrm{Div}\,\mathbf{S} + \rho_r\mathbf{b} = \rho_r\mathbf{a} \equiv \rho_r\mathbf{x}_{,tt}, \tag{6.10}$$

where $\mathbf{S}$ is given by $(2.83)_1$ for an unconstrained material and $(2.85)_1$ for an incompressible material, with $\mathbf{F}$ and p now depending on t. Equation (6.10) is an equation for the motion (6.2) subject to appropriate boundary and initial conditions, which are not listed here. In components equation (6.10) has the forms

$$\mathcal{A}^1_{\alpha i\beta j}x_{j,\alpha\beta} + \rho_r b_i = \rho_r x_{i,tt} \tag{6.11}$$

and

$$\mathcal{A}^1_{\alpha i\beta j}x_{j,\alpha\beta} - p_{,i} + \rho_r b_i = \rho_r x_{i,tt}, \quad \det(x_{i,\alpha}) = 1 \tag{6.12}$$

for unconstrained and incompressible materials respectively.

There are very few exact finite amplitude solutions of the dynamic equations (6.11) or (6.12) available in the literature. Some of these are outlined in the text by Eringen and Suhubi (1974) and we refer to this work for references up to the date of its publication. We mention the analysis of radial oscillations of a cylindrical tube by Knowles (1960, 1962), radial oscillations of a spherical shell by Guo and Solecki (1963a, b) and of a spherical cavity in an infinite medium by Knowles and Jakub (1965). More recent work includes the derivation of exact solutions by Rajagopal *et al.* (1989), Boulanger and Hayes (1989), Hayes and Rajagopal (1992) and Andreadou *et al.* (1993). For discussion of the propagation of finite amplitude waves we refer to, for example, Carroll (1978), Boulanger and Hayes (1992) and Boulanger *et al.* (1994). Further references are contained in these papers. Some aspects of nonlinear elastic wave propagation are discussed in Chapter 11 in this volume.

1.6.3 Incremental motions

We now consider incremental motions superimposed on a finite motion. Let

$$\dot{\mathbf{x}} = \dot{\chi}(\mathbf{X}, t) \tag{6.13}$$

be the dynamic counterpart of the increment defined in (4.1). Then, the dynamic counterpart of the incremental equilibrium equation (4.14) is

$$\operatorname{Div} \dot{\mathbf{S}} + \rho_r \dot{\mathbf{b}} = \rho_r \dot{\mathbf{x}}_{,tt}, \tag{6.14}$$

where a superposed dot again signifies an incremental quantity. In (6.14) no approximation has been made. However, when the equation is linearized in the incremental quantities it becomes, for an unconstrained material,

$$\operatorname{Div} (\boldsymbol{\mathcal{A}}^1 \dot{\mathbf{F}}) + \rho_r \dot{\mathbf{b}} = \rho_r \dot{\mathbf{x}}_{,tt}, \tag{6.15}$$

where $\dot{\mathbf{b}}$ has been linearized in $\dot{\mathbf{x}}$.

In components equation (6.15) may be written

$$\mathcal{A}^1_{\alpha i \beta j} \dot{x}_{j,\alpha\beta} + \mathcal{A}^2_{\alpha i \beta j \gamma k} \dot{x}_{j,\beta} x_{k,\alpha\gamma} + \rho_r \dot{b}_i = \rho_r \dot{x}_{i,tt}, \tag{6.16}$$

where $\mathcal{A}^2_{\alpha i \beta j \gamma k}$ are the components of the (sixth-order) tensor $\boldsymbol{\mathcal{A}}^2$ of second-order moduli defined by

$$\boldsymbol{\mathcal{A}}^2 = \frac{\partial^3 W}{\partial \mathbf{F}^3}, \quad \mathcal{A}^2_{\alpha i \beta j \gamma k} = \frac{\partial^3 W}{\partial F_{i\alpha} \partial F_{j\beta} \partial F_{k\gamma}}. \tag{6.17}$$

For the special case in which the incremental motion is superimposed on a *homogeneous static* finite deformation and body forces are omitted, equation (6.16) reduces to

$$\mathcal{A}^1_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} = \rho_r \dot{x}_{i,tt}, \tag{6.18}$$

where the coefficients $\mathcal{A}^1_{\alpha i\beta j}$ are now constants. This equation may be updated to the static finitely-deformed configuration to give

$$\mathcal{A}^1_{0piqj}u_{j,pq} = \rho u_{i,tt}, \tag{6.19}$$

where

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\chi(\mathbf{X}, t), t) \equiv \dot{\chi}(\mathbf{X}, t). \tag{6.20}$$

The counterpart of (6.19) for an incompressible material is

$$\mathcal{A}^1_{0piqj}u_{j,pq} - \dot{p}_{,i} = \rho u_{i,tt}, \quad u_{i,i} = 0. \tag{6.21}$$

Many specific problems have been examined on the basis of the incremental equations and we mention here just a selection of these. The problem of surface (Rayleigh) waves propagating on a homogeneously pre-strained half-space, the dynamic counterpart of the problem discussed in Section 1.5.1.1, was examined by Hayes and Rivlin (1961a), Flavin (1963) and Chadwick and Jarvis (1979). It was reconsidered from the point of view of its connection with stability of the underlying finite deformation by Dowaikh and Ogden (1990, 1991) for incompressible and unconstrained materials respectively. The surface wave problem for an underlying deformation corresponding to simple shear was examined by Connor and Ogden (1995). Wave propagation in a pre-strained plate was discussed by Ogden and Roxburgh (1993), Roxburgh and Ogden (1994) and Connor and Ogden (1996), again with reference to the underlying stability problem. References to related work can be found in these papers. References to papers dealing with wave propagation in pre-stressed cylinders are given by Eringen and Suhubi (1974) and Haughton (1982), but there is very little in the literature for problems with other underlying geometries.

1.6.3.1 Incremental plane waves

In this final section we consider the propagation of plane waves given in the form

$$\mathbf{u} = \mathbf{m} f(\mathbf{n}\cdot\mathbf{x} - ct), \tag{6.22}$$

where the unit vector $\mathbf{m}$ is the *polarization vector*, c is the *wave speed* and f is a twice continuously differentiable function. The unit vector $\mathbf{n}$, when real, defines the *direction of propagation* of the wave, which is then a *homogeneous plane wave*; in general $\mathbf{n}$ may be complex, in which case the wave is referred to as an *inhomogeneous plane wave.* In many applications f is taken to be an exponential function, but this is not necessary in general.

Substitution of (6.22) into the equation of motion (6.19) and (6.21) yields, after some manipulations,

$$\mathbf{Q}_0(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \tag{6.23}$$

and

$$[\mathbf{Q}_0(\mathbf{n}) - \mathbf{n} \otimes \mathbf{Q}_0(\mathbf{n})\mathbf{n}]\mathbf{m} = \rho c^2 \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{6.24}$$

respectively, where the tensor $\mathbf{Q}_0(\mathbf{n})$, which depends on $\mathbf{n}$, is defined (in component form) by

$$[\mathbf{Q}_0(\mathbf{n})]_{ij} = \mathcal{A}^1_{0piqj} n_p n_q. \tag{6.25}$$

In view of its connection with wave propagation $\mathbf{Q}_0(\mathbf{n})$ is called the *acoustic tensor.*

Equations (6.23) and (6.24), for unconstrained and incompressible materials respectively, are referred to as *propagation conditions.* They determine possible wave speeds and polarization vectors for which plane waves with a given direction $\mathbf{n}$ can propagate. The wave speeds are obtained as solutions of the *characteristic equation*

$$\det[\mathbf{Q}_0(\mathbf{n}) - \rho c^2 \mathbf{I}] = 0 \tag{6.26}$$

(for an unconstrained material), or

$$\det[\mathbf{Q}_0(\mathbf{n}) - \mathbf{n} \otimes \mathbf{Q}_0(\mathbf{n})\mathbf{n} - \rho c^2 \mathbf{I}] = 0 \tag{6.27}$$

(for an incompressible material), $\mathbf{I}$ being the identity tensor.

From either (6.23) or (6.24) we obtain

$$\rho c^2 = [\mathbf{Q}_0(\mathbf{n})\mathbf{m}] \cdot \mathbf{m}. \tag{6.28}$$

By expressing the right-hand side of (6.28) in component form it is then apparent from (2.127) that $\rho c^2 > 0$ follows from the strong ellipticity condition, and this provides an interpretation of the latter condition in terms of wave propagation. For further discussion of this connection and references see Ogden (1997).

The propagation of infinitesimal plane waves in a homogeneously deformed elastic material was first considered by Hayes and Rivlin (1961b). Recently, an extensive analysis of the reflection of plane waves at the boundary of a prestressed half-space subject to pure homogeneous strain has been carried out by Ogden and Sotiropoulos (1997, 1998) for incompressible and unconstrained materials respectively, and a corresponding analysis for a half-space subject to a simple shear deformation by Hussain and Ogden (1999). References to related works are contained in these papers.

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