Nonlinear Elasticity, Anisotropy, Material Stability and Residual stresses in Soft Tissue

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Abstract. In this chapter the basic equations of nonlinear elasticity theory needed for the analysis of the elastic behaviour of soft tissues are summarized. Particular attention is paid to characterizing the material symmetries associated with the anisotropy that arises in soft tissue from its fibrous constituents (collagens) that endow the material with preferred directions. The importance of the issue of convexity in the construction of constitutive laws (strain-energy functions) for soft tissues is emphasized with reference to material stability. The problem of extension and inflation of a thick-walled circular cylindrical tube is used throughout as an example that is closely associated with arterial wall mechanics. This is discussed first for isotropic materials, then for cylindrically orthotropic materials. Since residual stresses have a very important role in, in particular, arterial wall mechanics these are examined in some detail. Specifically, for the tube extension/inflation problem the residual stresses arising from the assumption that the circumferential stress is uniform under typical physiological conditions are calculated for a representative constitutive law and compared with those calculated using the 'opening angle' method.

1 Nonlinear Elasticity

The mathematical framework for describing the mechanical behaviour of biological soft tissues has much in common with that used in rubber elasticity, but there are significant differences in the structures of these materials and in the way that soft tissues and rubber respond under applied stresses. Figure 1 compares, for example, the typical simple tension stress-stretch response of rubber (left-hand figure) with that of soft tissue. An important characteristic of soft tissues is the initial large extension achieved with relatively low levels of stress and the subsequent stiffening at higher levels of extension, this being associated with the recruitment of collagen fibres as they become uncrimped and reach their natural lengths, whereupon their significant stiffness comes into play and overrides that of the underlying matrix material. The distribution of collagen fibres leads to the pronounced anisotropy in soft tissues, which distinguishes them from the typical (isotropic) rubber. For a detailed discussion of the morphological structure (histology) of arteries and relevant references we refer to Humphrey (1995), Holzapfel et al. (2000) and Holzapfel (2001a, b).

A more detailed picture of the response of soft tissue is illustrated in Figure 2. This shows results from *in vitro* experiments on a human iliac artery which is subjected to extension and inflation. Figure 2(a) shows a plot of the internal pressure against the circumferential stretch λ_θ for a series of fixed values of the (reduced) axial load. The curves show the typical stiffening referred to above. In Figure 2(b) the pressure is plotted against the axial stretch λ_z, again for

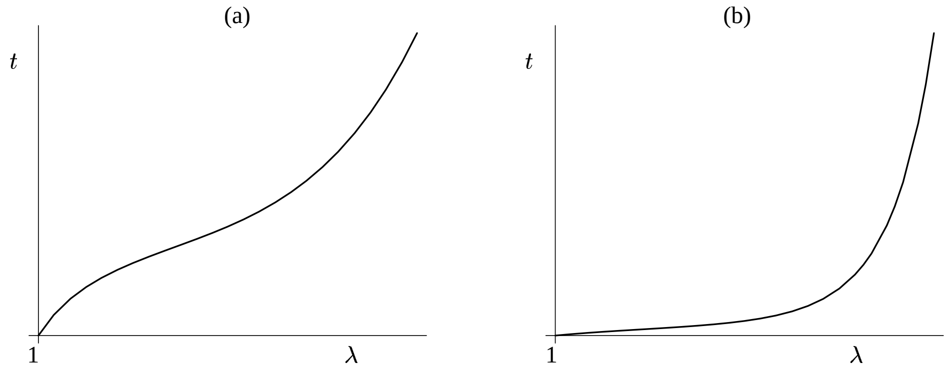

Figure 1. Typical simple tension response of (a) rubber and (b) soft tissue. Nominal stress $t \geq 0$ plotted against stretch $\lambda \geq 1$.

fixed values of the axial load. An interesting feature here is that a transitional value of the axial stretch (approximately 1.53) is identified, which is the *in vivo* value of the axial stretch. This value is unaffected by changes in pressure and corresponds to an axial load of 0.99 N. For lower (higher) values of the stretch the pressure stretch curves have a positive (negative) gradient, and the transitional value may therefore be referred to as an *inversion point*.

The data in Figure 2 are from an external iliac artery of a 52 year old female without any cardiovascular risk factors. The artery was healthy in the sense that it was not stenotic. By contrast, data from the other specimens considered by Schulze-Bauer et al. (2001), from individuals ranging from 57 to 87 years old (with cardiovascular risk factors), have qualitative characteristics similar to those shown in Figure 2, but there are some numerical differences. In particular, the value of the axial stretch at the inversion point is typically between 1.07 and 1.25, significantly less than the value seen in Figure 2.

Generally, soft tissue material can be regarded as incompressible and anisotropic. The degree and type of anisotropy depends very much on the tissue considered and its topographic location. Some tissues (such as tendon and ligament) are transversely isotropic and others (such as arteries) orthotropic, for example. Moreover, their stress-strain response is highly nonlinear, with the typical rapid stiffening with pressure arising from the recruitment of collagen fibrils and giving rise to the markedly anisotropic behaviour (Nichols and O'Rourke, 1998), as is illustrated in Figure 2. In some situations the mechanical response can be treated as purely *elastic*. For example, the *passive* behaviour of large *proximal* (close to the heart) arteries such as the aorta, and the iliac arteries can be regarded as essentially elastic, while the response of *distal* arteries, on the other hand, is viscoelastic. These notes, however, are concerned solely with elasticity and its application to some basic problems and geometries relevant to the characterization of the elastic response of soft tissues. The theory applies to many different soft tissues, but, for purposes of illustration, we confine much of the discussion to the analysis of the elasticity of arteries.

In this first section we summarize the basic equations and notation of nonlinear elasticity theory necessary for the continuum description of the mechanical properties of soft tissues, and we

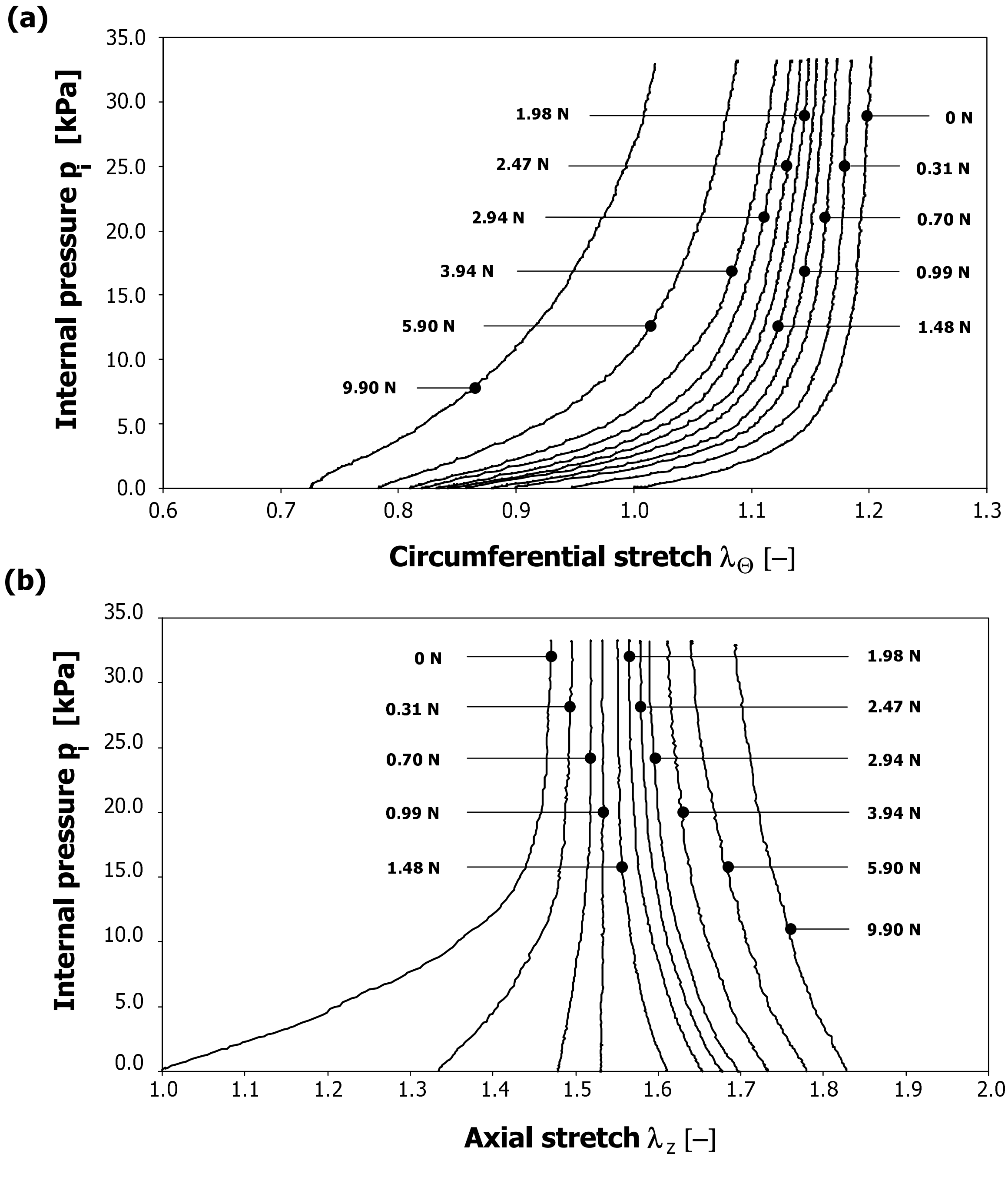

Figure 2. Typical characteristics of the response of a human iliac artery under pressure and axial load (Schulze-Bauer et al., 2001). Dependence of the internal pressure P_i kPa on (a) the circumferential stretch and (b) the axial stretch at values of the axial load up to 9.9N held constant during the deformation process.

examine some illustrative examples of elastic deformations for incompressible isotropic elastic materials. For a more detailed account we refer to, for example, Ogden (1997, 2001), Holzapfel (2000) and Fu and Ogden (2001). See, also, the chapter by Holzapfel (2001b) in this volume.

1.1 Kinematics

Let **X** and **x**, respectively, denote the position vector of a material point in some reference configuration, denoted $\mathcal{B}_r$, and the (deformed) current configuration, denoted $\mathcal{B}$, which may vary with time t. The *motion* (or *time-dependent deformation*) from $\mathcal{B}_r$ to $\mathcal{B}$ is known when **x** is specified as a function of **X** and t, and we write this in the form

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \tag{1}$$

where $\boldsymbol{\chi}$ is the function describing the motion. For each t, $\boldsymbol{\chi}$ is invertible and satisfies appropriate regularity conditions.

The *deformation gradient tensor*, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} \tag{2}$$

and has Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, where Grad is the gradient operator in $\mathcal{B}_r$ and x_i and X_α are the components of **x** and **X**, respectively, $i, \alpha \in \{1, 2, 3\}$. Local invertibility of the deformation requires that $\mathbf{F}$ be non-singular, and the usual convention that

$$J \equiv \det \mathbf{F} > 0 \tag{3}$$

is adopted, wherein the notation J is defined.

The velocity **v** and acceleration **a** of a material particle are given, respectively, by

$$\mathbf{v} = \frac{\partial \boldsymbol{\chi}}{\partial t}(\mathbf{X}, t), \quad \mathbf{a} = \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2}(\mathbf{X}, t), \tag{4}$$

these being the first and second material time derivatives of $\boldsymbol{\chi}$.

The (unique) *polar decompositions*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{5}$$

then follow, $\mathbf{R}$ being a proper orthogonal tensor and $\mathbf{U}$, $\mathbf{V}$ positive definite and symmetric tensors (the *right* and *left stretch tensors*, respectively). The tensors $\mathbf{U}$ and $\mathbf{V}$ have the *spectral decompositions*

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{6}$$

respectively, where $\lambda_i > 0$, $i \in \{1, 2, 3\}$, are the *principal stretches*, $\mathbf{u}^{(i)}$ are the (unit) eigenvectors of $\mathbf{U}$, called the *Lagrangian principal axes*, $\mathbf{v}^{(i)}$, the (unit) eigenvectors of $\mathbf{V}$, called the *Eulerian principal axes*, and $\otimes$ denotes the tensor product.

The *left* and *right Cauchy-Green deformation tensors*, denoted respectively by **B** and **C**, are defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \equiv \mathbf{V}^2, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} \equiv \mathbf{U}^2, \tag{7}$$

and the principal invariants of **B** (equivalently of **C**) are defined by

$$I_1 = \mathrm{tr}(\mathbf{B}), \quad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}(\mathbf{B}^2)], \quad I_3 = \det(\mathbf{B}) \equiv (\det \mathbf{F})^2. \tag{8}$$

The Green strain tensor, defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \tag{9}$$

where **I** is the identity tensor, will be required in the following sections.

If the material is incompressible, which is usually taken to be the case both for rubber and for soft tissues, then the *incompressibility constraint*

$$J \equiv \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 = 1, \qquad I_3 = 1 \tag{10}$$

must be satisfied.

1.2 Stress and the equations of motion

The equation of motion may be expressed in the form

$$\mathrm{Div}\, \mathbf{S} = \rho_r\, \mathbf{a} \equiv \rho_r\, \mathbf{x}_{,tt}, \tag{11}$$

where **S** is the nominal stress tensor (the transpose of the first Piola-Kirchhoff stress tensor), ρ_r is the mass density of the material in $\mathcal{B}_r$ and $_{,t}$ signifies the material time derivative. Body forces have been omitted. Equation (11) is the Lagrangian version of the equation of motion, with $(\mathbf{X}, t)$ used as the independent variables.

The global counterpart of the local balance equation (11) may be written

$$\int_{\partial \mathcal{B}_r} \mathbf{S}^T \mathbf{N} \mathrm{d}A = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{B}_r} \rho_r \mathbf{v} \mathrm{d}V, \tag{12}$$

where **N** is the unit outward normal to the boundary $\partial\mathcal{B}_r$ of $\mathcal{B}_r$, $\mathrm{d}A$ is the area element on $\partial\mathcal{B}_r$ and $\mathrm{d}V$ the volume element in $\mathcal{B}_r$. This serves to identify the *traction vector* $\mathbf{S}^T\mathbf{N}$ per unit area of $\partial\mathcal{B}_r$ (also referred to as the *load* or *stress vector*).

The equation of motion (11) may, equivalently, be written in the Eulerian form

$$\mathrm{div}\, \boldsymbol{\sigma} = \rho\, \mathbf{a}, \tag{13}$$

where the symmetric tensor $\boldsymbol{\sigma}$ is the Cauchy stress tensor, div is the divergence operator with respect to **x** and ρ is the material density in $\mathcal{B}$, with $(\mathbf{x}, t)$ as the independent variables and **a** treated, through the inverse of (1), as a function of $\mathbf{x}$ and t.

The two densities are connected through

$$\rho_r = \rho J, \tag{14}$$

so that if the material is incompressible then $\rho = \rho_r$.

The nominal and Cauchy stress tensors are related by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{15}$$

The *Biot stress tensor*, denoted **T**, and the *second Piola-Kirchhoff stress tensor*, denoted $\boldsymbol{\tau}$, which are both symmetric, will also be used in what follows. They are defined by

$$\mathbf{T} = \frac{1}{2}(\mathbf{SR} + \mathbf{R}^T\mathbf{S}^T), \quad \boldsymbol{\tau} = \mathbf{SF}^{-T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} \tag{16}$$

and connected through

$$\mathbf{T} = \frac{1}{2}(\boldsymbol{\tau}\mathbf{U} + \mathbf{U}\boldsymbol{\tau}). \tag{17}$$

1.3 Hyperelasticity

We consider an elastic material for which the material properties are characterized in terms of a *strain-energy function* (per unit volume), denoted $W = W(\mathbf{F})$ and defined on the space of deformation gradients. This theory is known as *hyperelasticity*. For an inhomogeneous material, i.e. one whose properties vary from point to point, W depends on **X** in addition to **F**, but we do not indicate this dependence explicitly in what follows.

For an unconstrained hyperelastic material the nominal stress is given by

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) \equiv \frac{\partial W}{\partial \mathbf{F}}, \tag{18}$$

wherein the notation **H** is defined. The tensor function **H** is referred to as the *response function of the material relative to the configuration* $\mathcal{B}_r$ in respect of the nominal stress tensor. In components, the derivative in (18) is written $S_{\alpha i} = \partial W/\partial F_{i\alpha}$, which provides our convention for ordering of the indices in the partial derivative with respect to **F**.

For an incompressible material the counterpart of (18) is

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{F}^{-1}, \qquad \det \mathbf{F} = 1, \tag{19}$$

where p is the Lagrange multiplier associated with the incompressibility constraint and is referred to as the *arbitrary hydrostatic pressure*.

The Cauchy stress tensor corresponding to (18), on use of (15), is then seen to be given by

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) \equiv J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \tag{20}$$

wherein the response function **G** associated with $\boldsymbol{\sigma}$ is defined. As for **H**, the form of **G** depends on the choice of reference configuration, and **G** is referred to as the *response function* of the material *relative* to $\mathcal{B}_r$ associated with the Cauchy stress tensor. Unlike **H**, however, **G** is a *symmetric* tensor-valued function. For incompressible materials (20) is replaced by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{I}, \qquad \det \mathbf{F} = 1. \tag{21}$$

If the configuration $\mathcal{B}_r$ is stress free then it is referred to as a *natural configuration*. Here, we take W and the stress to vanish in $\mathcal{B}_r$, so that, for an unconstrained material,

$$W(\mathbf{I}) = 0, \qquad \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{O}, \tag{22}$$

with appropriate modifications in the case of an incompressible material. If the stress does not vanish in $\mathcal{B}_r$ then this configuration is said to be *residually stressed*. In such a configuration the traction must vanish at all points of the boundary, so that, *a fortiori*, residual stress is inhomogeneous in character (i.e., it cannot be uniform). Residual stresses are important in the context of biological tissues and the restriction (22) will be removed later, in Section 4, where we consider the consequences of residual stress for the constitutive law and response of the material.

Objectivity The elastic stored energy is required to be independent of superimposed rigid motions of the form

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{23}$$

where **Q** is a proper orthogonal (rotation) tensor **c** is a translation vector. The resulting deformation gradient is **QF** and it therefore follows that

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{24}$$

for *all* rotations **Q**. A strain-energy function satisfying this requirement is said to be *objective*.

Use of the polar decomposition (5) and the choice $\mathbf{Q} = \mathbf{R}^T$ in (24) shows that

$$W(\mathbf{F}) = W(\mathbf{U}). \tag{25}$$

Thus, W depends on **F** only through the stretch tensor **U** and may therefore be defined on the class of positive definite symmetric tensors. Equivalently, through (7) and (9), W may be regarded as a function of the Green strain **E**.

Expressions analogous to (18) and (19) can therefore be written down for the Biot and second Piola-Kirchhoff stresses. Thus,

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}}, \quad \boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{E}}, \tag{26}$$

and

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} - p\,\mathbf{U}^{-1}, \quad \det \mathbf{U} = 1, \qquad \boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{E}} - p\,\mathbf{C}^{-1}, \quad \det \mathbf{C} = 1 \tag{27}$$

for unconstrained and incompressible materials respectively, where $\mathbf{C} = \mathbf{I} + 2\mathbf{E}$. Note that when expressed as a function of **U** or **E** the strain energy automatically satisfies the objectivity requirement.

Material symmetry Mathematically, there is no restriction so far other than (22) and (24) on the form that the function W may take. However, the predicted stress-strain behaviour based on the form of W must on the one hand be acceptable for the description of the elastic behaviour of real materials and on the other hand make mathematical sense.

Further restrictions on the form of W arise if the material possesses symmetries in the configuration $\mathcal{B}_r$. Material symmetry (relative to a given reference configuration) is identified by transformations of the reference configuration that do not affect the material response. Consider a change from the reference configuration $\mathcal{B}_r$ (in which material points are identified by position vectors **X**) to a new reference configuration $\mathcal{B}'_r$ (with material points identified by $\mathbf{X}'$), let $\mathbf{F}'$ denote the deformation gradient relative to $\mathcal{B}'_r$ and let $\mathrm{Grad}\,\mathbf{X}'$ be denoted by **P**. If the material response is unchanged then

$$W(\mathbf{F}'\mathbf{P}) = W(\mathbf{F}') \tag{28}$$

for *all* deformation gradients $\mathbf{F}'$. This states that the strain-energy function is unaffected by a change of reference configuration with deformation gradient **P**. The collection of **P** for which (28) holds forms a group, which is called the *symmetry group of the material relative to* $\mathcal{B}_r$. As already mentioned biological soft tissues are distinguished, in particular, by the anisotropy of their structure. We will examine the appropriate type of anisotropy in Section 2, but initially we shall, for simplicity, focus on the development of the theory in the case of isotropy.

Isotropy To be specific we now consider *isotropic elastic materials*, for which the symmetry group is the *proper orthogonal group*. Then, we have

$$W(\mathbf{FQ}) = W(\mathbf{F}) \tag{29}$$

for *all* rotations $\mathbf{Q}$. Bearing in mind that the $\mathbf{Q}$'s appearing in (24) and (29) are independent the combination of these two equations yields

$$W(\mathbf{QUQ}^T) = W(\mathbf{U}) \tag{30}$$

for all rotations $\mathbf{Q}$, or, equivalently, $W(\mathbf{QVQ}^T) = W(\mathbf{V})$. Equation (30) states that W is an *isotropic function* of $\mathbf{U}$. It follows from the spectral decomposition (5) that W depends on $\mathbf{U}$ only through the principal stretches $\lambda_1, \lambda_2, \lambda_3$. To avoid introducing additional notation we express this dependence as $W(\lambda_1, \lambda_2, \lambda_3)$; by selecting appropriate values for $\mathbf{Q}$ in (30) we may deduce that W depends symmetrically on $\lambda_1, \lambda_2, \lambda_3$, i.e.

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_2, \lambda_1, \lambda_3). \tag{31}$$

A consequence of isotropy is that the Biot stress $\mathbf{T}$ is *coaxial* with $\mathbf{U}$ and, equivalently, the Cauchy stress $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$. Hence, in parallel with (6), we have

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \qquad \boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{32}$$

where t_i, are the *principal Biot stresses* and σ_i are the *principal Cauchy stresses*. For an unconstrained material,

$$t_i = \frac{\partial W}{\partial \lambda_i}, \qquad J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \tag{33}$$

while for an incompressible material these are replaced by

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1}, \qquad \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad \lambda_1 \lambda_2 \lambda_3 = 1. \tag{34}$$

Note that in (33) and (34) there is no summation over the repeated index i.

For an isotropic material the symmetric dependence (31) of W on the principal stretches is equivalent to W being regarded as a function of the (symmetric) principal invariants I_1, I_2, I_3 defined by (8). In terms of the invariants I_1, I_2, I_3 the Cauchy stress tensor for an unconstrained isotropic elastic material may be written

$$\boldsymbol{\sigma} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2, \tag{35}$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ are functions of I_1, I_2, I_3 given by

$$\alpha_0 = 2I_3^{1/2}\frac{\partial W}{\partial I_3}, \quad \alpha_1 = 2I_3^{-1/2}\Big(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\Big), \quad \alpha_2 = -2I_3^{-1/2}\frac{\partial W}{\partial I_2}, \tag{36}$$

with W is now regarded as a function of I_1, I_2, I_3. For an incompressible material the corresponding expression is

$$\boldsymbol{\sigma} = -p\mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2, \tag{37}$$

where p is the arbitrary hydrostatic pressure, α_1 and α_2 are again given by (36) (but with $I_3 = 1$), and W is now regarded as a function of I_1 and I_2 alone.

1.4 Initial-boundary-value problems

For an unconstrained material we now consider the equation of motion (11) together with the stress-deformation relation (18), and the deformation gradient (2) coupled with (1). Thus,

$$\mathrm{Div}\left(\frac{\partial W}{\partial \mathbf{F}}\right) = \rho_r \mathbf{x}_{,tt}, \quad \mathbf{F} = \mathrm{Grad}\,\mathbf{x}, \quad \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad \mathbf{X} \in \mathcal{B}_r. \tag{38}$$

An initial-boundary-value problem is obtained by supplementing (38) with appropriate boundary and initial conditions.

Typical boundary conditions arising in problems of nonlinear elasticity are those in which $\mathbf{x}$ is specified on part of the boundary, $\partial\mathcal{B}_r^x \subset \partial\mathcal{B}_r$ say, and the stress vector on the remainder, $\partial\mathcal{B}_r^s$, so that $\partial\mathcal{B}_r^x \cup \partial\mathcal{B}_r^s = \partial\mathcal{B}_r$ and $\partial\mathcal{B}_r^x \cap \partial\mathcal{B}_r^s = \emptyset$. We write

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}, t) \quad \text{on } \partial\mathcal{B}_r^x, \tag{39}$$

$$\mathbf{S}^T\mathbf{N} = \mathbf{s}(\mathbf{F}, \mathbf{X}, t) \quad \text{on } \partial\mathcal{B}_r^s, \tag{40}$$

where $\boldsymbol{\xi}$ and $\mathbf{s}$ are specified functions. In general, $\mathbf{s}$ may depend on the deformation and this is indicated in (40) by the explicit dependence of $\mathbf{s}$ on the deformation gradient **F**. If, for example, the boundary traction in (40) is associated with a hydrostatic pressure, P say, so that $\boldsymbol{\sigma}\mathbf{n} = -P\mathbf{n}$, where **n** is the unit outward normal to $\partial\mathcal{B}$, then $\mathbf{s}$ depends on the deformation in the form

$$\mathbf{s} = -JP\mathbf{F}^{-T}\mathbf{N} \quad \text{on } \partial\mathcal{B}_r^s. \tag{41}$$

We record here that **n** and **N** are related through Nanson's formula $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A$, where $\mathrm{d}a$ is the area element on $\partial\mathcal{B}$. In the important special case in which the surface traction defined by (40) is independent of **F** and t it is referred to as a *dead-load traction*.

A variety of possible initial conditions may be specified. A typical set corresponds to prescription of the initial values of $\mathbf{x}$ and $\mathbf{x}_{,t}$. Thus,

$$\boldsymbol{\chi}(\mathbf{X},0)=\mathbf{x}_0(\mathbf{X}), \quad \frac{\partial\boldsymbol{\chi}}{\partial t}(\mathbf{X},0)=\mathbf{v}_0(\mathbf{X}) \text{ on } \mathcal{B}_r, \tag{42}$$

where $\mathbf{x}_0$ and $\mathbf{v}_0$ are prescribed functions. A basic initial-boundary-value problem of nonlinear elasticity is then characterized by (38)–(40) with (42).

In components, the equation of motion in (38) can be written

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta}=\rho_r\, x_{i,tt}, \tag{43}$$

for $i\in\{1,2,3\}$, where the coefficients $\mathcal{A}_{\alpha i\beta j}$ are defined by

$$\mathcal{A}_{\alpha i\beta j}=\mathcal{A}_{\beta j\alpha i}=\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}. \tag{44}$$

The pairwise symmetry of the indices in (44) should be noted.

For *incompressible materials* the corresponding equations, obtained by substituting $(19)_1$ into (11), are

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta}-\frac{\partial p}{\partial x_i}=\rho_r\, x_{i,tt} \tag{45}$$

subject to (10), where the coefficients are again given by (44).

The coefficients $\mathcal{A}_{\alpha i\beta j}$ are, in general, nonlinear functions of the components of the deformation gradient. Explicit expressions for the components $\mathcal{A}_{\alpha i\beta j}$ in respect of an isotropic material can be found, for example, in Ogden (1997), and we shall make use of them in Section 3. We emphasize that in the above Cartesian coordinates are being used. In Section 1.5.2 and subsequently we shall require the radial equation in cylindrical polar coordinates. This will be given at the appropriate point. More general expressions for the (equilibrium) equations in cylindrical polar coordinates and other coordinate systems are given in Ogden (1997), for example.

In order to analyze such initial-boundary-value problems additional information about the nature of the function W is required. This information may come from the construction of special forms of strain-energy function based on comparison of theory with experiment for particular materials, it may arise naturally in the course of solution of particular problems, from considerations of stability for example, or may be derived from mathematical requirements on the properties that W should possess in order for existence of solutions to be guaranteed, for example. Consideration of the form of W is therefore absolutely crucial, and some aspects of this will be examined in the course of the following sections.

Finally in this section, we give an expression for the coefficients $\mathcal{A}_{\alpha i\beta j}$ in terms of the derivatives of W with respect to the Green strain $\mathbf{E}$ since the latter (or, equivalently, $\mathbf{C}$) is often used in computational formulations of the equations of motion (or equilibrium). This is

$$\mathcal{A}_{\alpha i\beta j}=F_{i\gamma}F_{j\delta}\frac{\partial^2 W}{\partial E_{\alpha\gamma}\partial E_{\beta\delta}}+\delta_{ij}\frac{\partial W}{\partial E_{\alpha\beta}}. \tag{46}$$

Note that for an incompressible material the form of these coefficients depends on the point at which the incompressibility condition is invoked in calculating the partial derivatives. This does not lead to any inconsistencies since differences are absorbed by the arbitrary hydrostatic pressure term. Expressions analogous to (46) may be written down for other strain or deformation measures such as **U** but we do not require them here.

1.5 Examples

Homogeneous deformations We consider first an elementary time-independent problem in which the deformation is *homogeneous*, so that the deformation gradient **F** is constant. Specifically, we consider the *pure homogeneous strain* defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{47}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are constants. For this deformation $\mathbf{F} = \mathbf{U} = \mathbf{V}, \mathbf{R} = \mathbf{I}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions and are fixed as the values of the stretches change. Thus, $\mathbf{F} \equiv \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. For an unconstrained isotropic elastic material the associated principal Biot stresses are given by $(33)_1$. These equations serve as a basis for determining the form of W from triaxial experimental tests in which $\lambda_1, \lambda_2, \lambda_3$ and t_1, t_2, t_3 are measured. If biaxial tests are conducted on a thin sheet of material which lies in the (X_1, X_2)-plane with no force applied to the faces of the sheet then equations $(33)_1$ reduce to

$$t_1 = \frac{\partial W}{\partial \lambda_1}(\lambda_1, \lambda_2, \lambda_3), \quad t_2 = \frac{\partial W}{\partial \lambda_2}(\lambda_1, \lambda_2, \lambda_3), \quad t_3 = \frac{\partial W}{\partial \lambda_3}(\lambda_1, \lambda_2, \lambda_3) = 0, \tag{48}$$

and the third equation gives λ_3 implicitly in terms of λ_1 and λ_2 when W is known.

The biaxial test is particularly important when the incompressibility constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{49}$$

holds since then only two stretches can be varied independently and biaxial tests are sufficient to obtain a characterization of W. The counterpart of (33) for the incompressible case is given by (34). It is convenient to make use of (49) to express the strain energy as a function of two independent stretches, and for this purpose we define

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}). \tag{50}$$

This enables p to be eliminated from equations (34) and leads to

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{51}$$

It is important to note that, because of the incompressibility constraint, equation (51) is unaffected by the superposition of an arbitrary hydrostatic stress. Thus, without loss of generality, we may set $\sigma_3 = 0$ in (51). In terms of the principal Biot stresses we then have simply

$$t_1 = \frac{\partial \hat{W}}{\partial \lambda_1}, \quad t_2 = \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{52}$$

which provides two equations relating λ_1, λ_2 and t_1, t_2 and therefore a basis for characterizing $\hat{W}$ from measured biaxial data. This situation is special for an isotropic material, and, as we shall see in Section 2, biaxial tests alone are not sufficient for the characterization of the strain-energy function of an anisotropic material.

There are several special cases of the biaxial test which are of interest, but we just give the details for one, namely *simple tension*, for which we set $t_2 = 0$. By symmetry, the incompressibility constraint then yields $\lambda_2 = \lambda_3 = \lambda_1^{-1/2}$. The strain energy may now be treated as a function of just λ_1, which we write as λ, and we define

$$\tilde{W}(\lambda) = \hat{W}(\lambda, \lambda^{-1/2}), \tag{53}$$

and $(52)_1$ reduces to

$$t = \tilde{W}'(\lambda), \tag{54}$$

where the prime indicates differentiation with respect to λ and t_1 has been replaced by t. It is the notation $(\lambda,\ t)$ that has been used in Figure 1.

For future reference we note here the form of (51) when W is regarded as a function of I_1 and I_2. Thus,

$$\sigma_1 - \sigma_3 = 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^4\lambda_2^2 - 1)\Big(\frac{\partial \check{W}}{\partial I_1} + \lambda_2^2 \frac{\partial \check{W}}{\partial I_2}\Big), \tag{55}$$

$$\sigma_2 - \sigma_3 = 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^2\lambda_2^4 - 1)\Big(\frac{\partial \check{W}}{\partial I_1} + \lambda_1^2 \frac{\partial \check{W}}{\partial I_2}\Big), \tag{56}$$

where $\check{W}(I_1, I_2) \equiv \hat{W}(\lambda_1, \lambda_2)$.

Extension and inflation of a thick-walled tube Here we examine a prototype example of a *non-homogeneous* deformation that is particularly relevant to the mechanics of arteries. We consider a thick-walled circular cylindrical tube whose initial geometry is defined by

$$A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{57}$$

where A, B, L are positive constants and R, Θ, Z are cylindrical polar coordinates. The tube is deformed so that the circular cylindrical shape is maintained, and the material of the tube is taken to be incompressible. The resulting deformation is then described by the equations

$$r^2 - a^2 = \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{58}$$

where r, θ, z are cylindrical polar coordinates in the deformed configuration, λ_z is the (uniform) axial stretch and a is the internal radius of the deformed tube.

The principal stretches $\lambda_1, \lambda_2, \lambda_3$ are associated respectively with the radial, azimuthal and axial directions and are written

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} = \lambda, \quad \lambda_3 = \lambda_z, \tag{59}$$

wherein the notation λ is introduced. It follows from (58) and (59) that

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{60}$$

where

$$\lambda_a = a/A, \quad \lambda_b = b/B. \tag{61}$$

Note that for given λ_z and A/B, λ_b depends on λ_a. For a fixed value of λ_z the inequalities

$$\lambda^2\lambda_z \geq 1, \quad \lambda_a \geq \lambda \geq \lambda_b \tag{62}$$

hold during inflation of the tube, with equality holding if and only if $\lambda = \lambda_z^{-1/2}$ for $A \leq R \leq B$.

We use the notation (50) for the strain energy but with $\lambda_2 = \lambda$ and $\lambda_3 = \lambda_z$ as the independent stretches, so that

$$\hat{W}(\lambda, \lambda_z) = W(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z). \tag{63}$$

Hence

$$\sigma_2 - \sigma_1 = \lambda\frac{\partial\hat{W}}{\partial\lambda}, \quad \sigma_3 - \sigma_1 = \lambda_z\frac{\partial\hat{W}}{\partial\lambda_z}. \tag{64}$$

Note that *locally* the deformation has the same structure as that of the pure homogeneous strain discussed in Section 1.5.1.

When there is no time dependence the equation of motion (13) reduces to the equilibrium equation $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$. For the considered symmetry its cylindrical polar component form reduces to the single scalar equation

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{65}$$

where $\sigma_{rr} = \sigma_1$, $\sigma_{\theta\theta} = \sigma_2$ in terms of the principal Cauchy stresses. This is to be solved in conjunction with the boundary conditions

$$\sigma_{rr} = \begin{cases} -P & \text{on } R = A \\ 0 & \text{on } R = B \end{cases} \tag{66}$$

corresponding to pressure $P\ (\geq 0)$ on the inside of the tube and zero traction on the outside.

On use of (58)–(61) the independent variable may be changed from r to λ, and integration of (65) and use of the boundary conditions (66) then yields

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\frac{\partial\hat{W}}{\partial\lambda}\mathrm{d}\lambda. \tag{67}$$

Since, from (60), λ_b depends on λ_a, equation (67) provides an expression for P as a function of λ_a (equivalently of the internal radius) when λ_z is fixed. In order to hold λ_z fixed an axial load, N say, must be applied to the ends of the tube. This is given by

$$N/\pi A^2 = (\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}\Big(2\lambda_z\frac{\partial\hat{W}}{\partial\lambda_z} - \lambda\frac{\partial\hat{W}}{\partial\lambda}\Big)\lambda\mathrm{d}\lambda + P\lambda_a^2. \tag{68}$$

Equation (68) applies to a tube with closed ends so that the pressure contributes to the axial load.

An illustrative plot of the pressure P calculated from (67) for a fixed value of λ_z ($= 1.2$) is shown in Figure 3(a) in dimensionless form in respect of the simple incompressible isotropic strain-energy function with $\hat{W}$ given by

$$\hat{W}(\lambda, \lambda_z) = \frac{2\mu}{n^2}(\lambda^n + \lambda_z^n + \lambda^{-n}\lambda_z^{-n} - 3), \tag{69}$$

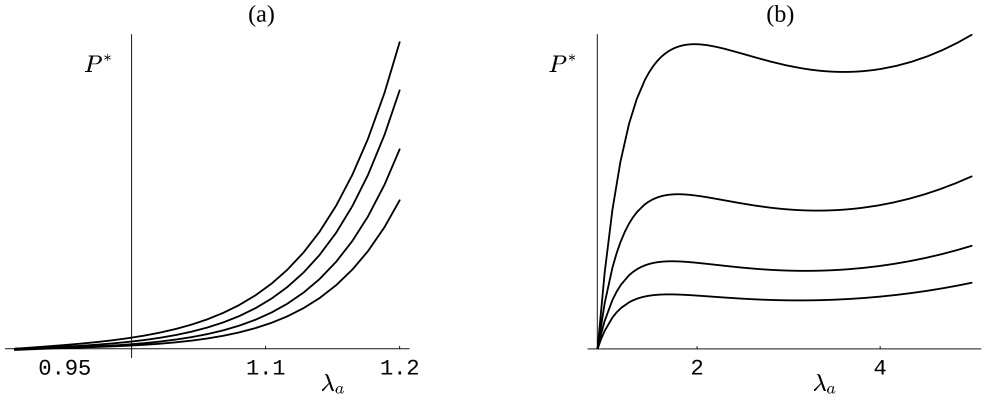

Figure 3. Plot of the dimensionless pressure P^* against stretch λ_a for different wall thicknesses and an axial pre-stretch $\lambda_z = 1.2$ in respect of (a) the strain-energy function (69) with $n = 24$, and (b) a typical rubberlike material.

where the constant μ is the shear modulus of the material and n is a dimensionless material constant. This is a special case of the class of strain-energy functions introduced by Ogden (1972). The dimensionless pressure is given by $P^* = nP/2\mu$ and results are shown for $n = 24$. In the figure results are compared for different values of the ratio A^2/B^2, specifically $0.4, 0.63, 0.77, 0.85$, reading from left to right in the figure. The feature to be noted here is the increasing stiffness with wall thickness. The results are contrasted with those for a typical rubberlike material shown in Figure 3(b) for the same values of A^2/B^2 (reading from top to bottom). The non-uniqueness in the relationship between P^* and λ_a evident in Figure 3(b) may be (but is not necessarily) associated with loss of stability of the symmetrical configuration and the possible emergence of bulges in the tube wall, a phenomenon which is excluded when P^* is a monotonically increasing function of λ_a, as is the situation in Figure 3(a). A detailed treatment of the inflation and extension of thick- and thin-walled tubes of rubber, with particular reference to loss of stability is given by Haughton and Ogden (1979a, b). For an account of constitutive laws for rubberlike solids we refer to Ogden (1982), while for a review that includes discussion of both elastomers and biological tissues the paper by Beatty (1987) is recommended.

2 Anisotropy

2.1 Fibre-reinforced materials

An important characteristic of soft tissues is the anisotropy in their mechanical properties that arises as a result of tensile stresses. This is associated with the collagen fibres that are distributed in the material and endow the material with directional properties. In some soft tissues (for example, ligament and tendon) this leads to the material having, in a macroscopic sense, a single preferred direction (see, for example, the discussion in Holzapfel, 2001a). The properties of the material can therefore be regarded as *transversely isotropic*. Other soft tissues, such as artery walls have two distinct distributions of collagen fibre directions and these can be associated with two preferred directions. Here, we illustrate the structure of the strain-energy function of an anisotropic elastic solid both for transverse isotropy and for its extension to the case of two preferred directions.

One family of fibres: transverse isotropy Firstly, we consider transverse isotropy. Let the unit vector **M** be a preferred direction in the reference configuration $\mathcal{B}_r$ of the material. The material response is then indifferent to arbitrary rotations about the direction **M** and by replacement of **M** by $-\mathbf{M}$. Such a material can be characterized with a strain energy which depends on **F** and the tensor $\mathbf{M}\otimes\mathbf{M}$, as described by Spencer (1972, 1984); see, also, Holzapfel (2000). Thus, we write $W(\mathbf{F},\mathbf{M}\otimes\mathbf{M})$ and, for an unconstrained material, the required symmetry reduces W to dependence on five invariants, namely $I_1,\ I_2,\ I_3$, as defined by (8), and the additional invariants I_4 and I_5 depending on **M** and defined by

$$I_4=\mathbf{M}\cdot(\mathbf{C}\mathbf{M}),\quad I_5=\mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}), \tag{70}$$

where **C** is the right Cauchy-Green deformation tensor defined in (7). We note that I_4 has a direct kinematical interpretation since $I_4^{1/2}$ represents the stretch in the direction **M**, but that there is no similar simple interpretation for I_5 in general.

On use of (18) the resulting nominal stress tensor is expressed in the form

$$\begin{aligned}\mathbf{S}=&\,2W_1\mathbf{F}^T+2W_2(I_1\mathbf{I}-\mathbf{C})\mathbf{F}^T+2I_3W_3\mathbf{F}^{-1}+2W_4\mathbf{M}\otimes\mathbf{F}\mathbf{M}\\&+2W_5(\mathbf{M}\otimes\mathbf{F}\mathbf{C}\mathbf{M}+\mathbf{C}\mathbf{M}\otimes\mathbf{F}\mathbf{M}),\end{aligned} \tag{71}$$

where $W_i=\partial W/\partial I_i, i=1,\dots,5$. For an isotropic material the terms in W_4 and W_5 are omitted. Equation (71) describes the response of a fibre-reinforced material with the fibre direction corresponding to **M** locally in the reference configuration. The vector field **M** can be thought of as a field that models the fibres as a continuous distribution.

For an incompressible material $I_3\equiv 1$, and the counterpart of (71) is

$$\begin{aligned}\mathbf{S}=&-p\,\mathbf{F}^{-1}+2\check{W}_1\mathbf{F}^T+2\check{W}_2(I_1\mathbf{I}-\mathbf{C})\mathbf{F}^T+2\check{W}_4\mathbf{M}\otimes\mathbf{F}\mathbf{M}\\&+2\check{W}_5(\mathbf{M}\otimes\mathbf{F}\mathbf{C}\mathbf{M}+\mathbf{C}\mathbf{M}\otimes\mathbf{F}\mathbf{M}),\end{aligned} \tag{72}$$

where, extending the notation defined following (56), the notation $\check{W}$ is used when W is regarded as a function of (I_1,I_2,I_4,I_5). The corresponding Cauchy stress tensor is given by

$$\begin{aligned}\boldsymbol{\sigma}=&-p\mathbf{I}+2\check{W}_1\mathbf{B}+2\check{W}_2(I_1\mathbf{B}-\mathbf{B}^2)+2\check{W}_4\mathbf{F}\mathbf{M}\otimes\mathbf{F}\mathbf{M}\\&+2\check{W}_5(\mathbf{F}\mathbf{M}\otimes\mathbf{B}\mathbf{F}\mathbf{M}+\mathbf{B}\mathbf{F}\mathbf{M}\otimes\mathbf{F}\mathbf{M}),\end{aligned} \tag{73}$$

where **B** is the left Cauchy-Green deformation tensor. The symmetry of $\boldsymbol{\sigma}$ can be seen immediately in (73), and we note that (73) reduces to (37) when the dependence on I_4 and I_5 is omitted.

For the pure homogeneous strain defined by (47) it is interesting to contrast the results for an isotropic material with those derived from (73). With respect to Cartesian axes, suppose that **M** has components $(\cos\varphi, \sin\varphi, 0)$, i.e. **M** lies in the principal plane associated with the stretches λ_1 and λ_2. Then, we have

$$I_4 = \lambda_1^2\cos^2\varphi + \lambda_2^2\sin^2\varphi, \quad I_5 = \lambda_1^4\cos^2\varphi + \lambda_2^4\sin^2\varphi, \tag{74}$$

and the connection

$$I_5 = I_4^2 + (\lambda_1^2 - \lambda_2^2)^2\sin^2\varphi\cos^2\varphi \tag{75}$$

is obtained in this case.

The components of $\boldsymbol{\sigma}$ are given by

$$\sigma_{11} = -p + 2\check{W}_1\lambda_1^2 + 2\check{W}_2\lambda_1^2(\lambda_2^2 + \lambda_3^2) + 2\check{W}_4\lambda_1^2\cos^2\varphi + 4\check{W}_5\lambda_1^4\cos^2\varphi, \tag{76}$$

$$\sigma_{22} = -p + 2\check{W}_1\lambda_2^2 + 2\check{W}_2\lambda_2^2(\lambda_1^2 + \lambda_3^2) + 2\check{W}_4\lambda_2^2\sin^2\varphi + 4\check{W}_5\lambda_2^4\sin^2\varphi, \tag{77}$$

$$\sigma_{12} = 2[\check{W}_4 + \check{W}_5(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\varphi\cos\varphi, \tag{78}$$

$$\sigma_{33} = -p + 2\check{W}_1\lambda_3^2 + 2\check{W}_2\lambda_3^2(\lambda_1^2 + \lambda_2^2), \quad \sigma_{13} = \sigma_{23} = 0. \tag{79}$$

Note that σ_{12} vanishes if the preferred direction is along one of the coordinate axes.

From (76), (77) and (79) it follows that

$$\begin{aligned}\sigma_{11} - \sigma_{33} = {}& 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^4\lambda_2^2 - 1)(\check{W}_1 + \lambda_2^2\check{W}_2)\\ &+ 2\check{W}_4\lambda_1^2\cos^2\varphi + 4\check{W}_5\lambda_1^4\cos^2\varphi,\end{aligned} \tag{80}$$

$$\begin{aligned}\sigma_{22} - \sigma_{33} = {}& 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^2\lambda_2^4 - 1)(\check{W}_1 + \lambda_1^2\check{W}_2)\\ &+ 2\check{W}_4\lambda_2^2\sin^2\varphi + 4\check{W}_5\lambda_2^4\sin^2\varphi,\end{aligned} \tag{81}$$

which extends the formulas (55) and (56) to the case of transverse isotropy.

Because of the incompressibility condition (49), I_1, I_2, I_4, I_5, and hence the strain-energy function, depend only on λ_1, λ_2 and the angle φ. In parallel with the definition (50) we therefore use the notation

$$\hat{W}(\lambda_1, \lambda_2, \varphi) = \check{W}(I_1, I_2, I_4, I_5) \tag{82}$$

to indicate this dependence. Note, however, that in general, in contrast to the isotropic situation, $\hat{W}(\lambda_1, \lambda_2, \varphi)$ is *not symmetric* in λ_1 and λ_2. It follows that (80) and (81) may be written simply in the form

$$\sigma_{11} - \sigma_{33} = \lambda_1\frac{\partial\hat{W}}{\partial\lambda_1}, \quad \sigma_{22} - \sigma_{33} = \lambda_2\frac{\partial\hat{W}}{\partial\lambda_2}. \tag{83}$$

These equations are very similar to equations (51), but here σ_{11} and σ_{22} are not principal stresses since the shear stress σ_{12} does not in general vanish (the principal axes of $\boldsymbol{\sigma}$ do not coincide with the principal axes of **B**, in contrast to the case for isotropy), and, it should be emphasized, that $\hat{W}(\lambda_1, \lambda_2, \varphi)$ is not symmetric in λ_1 and λ_2. The shear stress σ_{12} is required to maintain the prescribed deformation. From (78) we see that the latter depends on the material properties through the dependence of W on I_4 and I_5 and (separately) on the value of φ. It is interesting to note that if φ is treated as a parameter then σ_{12} can be expressed in the form

$$\sigma_{12} = \frac{\lambda_1\lambda_2}{\lambda_2^2 - \lambda_1^2}\frac{\partial\hat{W}}{\partial\varphi}. \tag{84}$$

It follows that, unlike the situation for incompressible isotropic materials, homogeneous biaxial deformations are not on their own sufficient to characterize the properties of a transversely isotropic elastic material.

Let **m** denote the unit vector in the direction **FM**. For the considered pure strain **FM** has components $(\lambda_1 \cos\varphi, \lambda_2 \sin\varphi, 0)$. Suppose that **m** has components $(\cos\varphi^*, \sin\varphi^*, 0)$. Then it follows that the angle φ^* is given by

$$\tan\varphi^* = \lambda_2\lambda_1^{-1}\tan\varphi. \tag{85}$$

Thus, in general, the orientation of the fibres is changed by the deformation.

Two families of fibres: orthotropy When there are two families of fibres corresponding to two preferred directions in the reference configuration, $\mathbf{M}$ and $\mathbf{M}'$ say, then, in addition to (8) and (70), the strain energy depends on the invariants

$$I_6 = \mathbf{M}' \cdot (\mathbf{CM}'),\ I_7 = \mathbf{M}' \cdot (\mathbf{C}^2\mathbf{M}'),\ I_8 = \mathbf{M} \cdot (\mathbf{CM}'), \tag{86}$$

and also on $\mathbf{M} \cdot \mathbf{M}'$ (which does not depend on the deformation); see Spencer (1972, 1984) for details. Note that I_8 involves interaction between the two preferred directions, but the term $\mathbf{M} \cdot (\mathbf{C}^2\mathbf{M}')$, which might be expected to appear in the list (86), is omitted since it depends on the other invariants and on $\mathbf{M} \cdot \mathbf{M}'$.

For a compressible material the nominal stress (71) is now extended to

$$\begin{aligned}\mathbf{S} = {} & 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T + 2I_3W_3\mathbf{F}^{-1} + 2W_4\mathbf{M}\otimes\mathbf{FM}\\ & + 2W_5(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}) + 2W_6\mathbf{M}'\otimes\mathbf{FM}'\\ & + 2W_7(\mathbf{M}'\otimes\mathbf{FCM}' + \mathbf{CM}'\otimes\mathbf{FM}')\\ & + W_8(\mathbf{M}\otimes\mathbf{FM}' + \mathbf{M}'\otimes\mathbf{FM}),\end{aligned} \tag{87}$$

where the notation $W_i = \partial W/\partial I_i$ now applies for $i = 1,\dots,8$. The counterpart of (87) for an incompressible material may be obtained on the same basis as for (72).

We now use the notation $\check{W}$ to represent W for an incompressible material when regarded as a function of $I_1, I_2, I_4, I_5, I_6, I_7, I_8$. The Cauchy stress tensor is then written

$$\begin{aligned}\boldsymbol{\sigma} = {} & -p\mathbf{I} + 2\check{W}_1\mathbf{B} + 2\check{W}_2(I_1\mathbf{B} - \mathbf{B}^2) + 2\check{W}_4\mathbf{FM}\otimes\mathbf{FM}\\ & + 2\check{W}_5(\mathbf{FM}\otimes\mathbf{BFM} + \mathbf{BFM}\otimes\mathbf{FM}) + 2\check{W}_6\mathbf{FM}'\otimes\mathbf{FM}'\\ & + 2\check{W}_7(\mathbf{FM}'\otimes\mathbf{BFM}' + \mathbf{BFM}'\otimes\mathbf{FM}')\\ & + \check{W}_8(\mathbf{FM}\otimes\mathbf{FM}' + \mathbf{FM}'\otimes\mathbf{FM}),\end{aligned} \tag{88}$$

the component form of which will be used in the following analysis.

For the solution of an initial-boundary-value problem on the basis of equations (38)–(42), for example, the form of the stress given by (87) or (88) is required. Clearly, these forms are very complicated and, in general, little can be achieved without the use of numerical computation, and this requires an appropriate choice of a particular form of $\check{W}$. A number of specific forms based on the use of the invariants discussed above have been used in the literature (see, for example, Humphrey, 1995, 1999, and Holzapfel *et al.*, 2000, for references). Other forms of anisotropic energy function not based directly on the use of these invariants have also been used in the biomechanics literature, as discussed in Humphrey (1995) and Holzapfel *et al.* (2000). Here, our intention is examine two specific and very simple deformations, namely the pure homogeneous strain and extension/inflation of a tube discussed (for an incompressible isotropic elastic material) in Sections 1.5.1 and 1.5.2 respectively, in respect of a general form of strain-energy function of the considered class in order to extract some qualitative and quantitative information about the nature of the energy function and its predictions.

Pure homogeneous strain. Again we consider the pure homogeneous strain defined by (47) and now we include two fibre directions, symmetrically disposed in the (X_1, X_2)-plane and given by

$$\mathbf{M} = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2, \quad \mathbf{M}' = \cos\varphi\,\mathbf{e}_1 - \sin\varphi\,\mathbf{e}_2, \tag{89}$$

where the angle φ is constant and $\mathbf{e}_1, \mathbf{e}_2$ denote the Cartesian coordinate directions. Let the corresponding unit vectors in the deformed configuration be denoted

$$\mathbf{m} = \cos\varphi^*\,\mathbf{e}_1 + \sin\varphi^*\,\mathbf{e}_2, \quad \mathbf{m}' = \cos\varphi^*\,\mathbf{e}_1 - \sin\varphi^*\,\mathbf{e}_2, \tag{90}$$

with φ^* again being given by (83). The deformation and fibre directions are depicted in Figure 4.

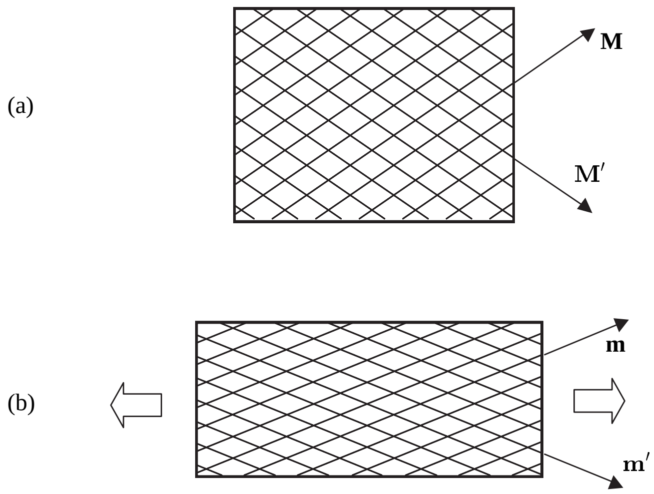

Figure 4. Pure homogeneous strain of a thin sheet of material with two in-plane symmetrically disposed families of fibres: (a) undeformed configuration; (b) deformed configuration.

When expressed in terms of λ_1 and λ_2 the invariants I_1, I_2 are given by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2, \tag{91}$$

while the other invariants are explicitly

$$I_4 = I_6 = \lambda_1^2 \cos^2\varphi + \lambda_2^2 \sin^2\varphi, \quad I_5 = I_7 = \lambda_1^4 \cos^2\varphi + \lambda_2^4 \sin^2\varphi, \tag{92}$$

$$I_8 = \lambda_1^2 \cos^2\varphi - \lambda_2^2 \sin^2\varphi. \tag{93}$$

From (88) the components of $\boldsymbol{\sigma}$ are found to be

$$\begin{aligned}\sigma_{11} &= -p + 2\check{W}_1\lambda_1^2 + 2\check{W}_2(I_1\lambda_1^2 - \lambda_1^4) + 2(\check{W}_4 + \check{W}_6 + \check{W}_8)\lambda_1^2\cos^2\varphi \\ &\quad + 4(\check{W}_5 + \check{W}_7)\lambda_1^4\cos^2\varphi, \end{aligned}\tag{94}$$

$$\begin{aligned}\sigma_{22} &= -p + 2\check{W}_1\lambda_2^2 + 2\check{W}_2(I_1\lambda_2^2 - \lambda_2^4) + 2(\check{W}_4 + \check{W}_6 - \check{W}_8)\lambda_2^2\sin^2\varphi \\ &\quad + 4(\check{W}_5 + \check{W}_7)\lambda_2^4\sin^2\varphi, \end{aligned}\tag{95}$$

$$\sigma_{12} = 2[\check{W}_4 - \check{W}_6 + (\check{W}_5 - \check{W}_7)(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\varphi\cos\varphi, \tag{96}$$

$$\sigma_{33} = -p + 2\check{W}_1\lambda_3^2 + 2\check{W}_2(I_1\lambda_3^2 - \lambda_3^4), \quad \sigma_{13} = \sigma_{23} = 0, \tag{97}$$

and we note that (97) is identical if form to (79) but is different in content since $\check{W}$ now depends on I_6, I_7, I_8.

In general, since $\sigma_{12} \neq 0$, shear stresses are required to maintain the pure homogeneous deformation and the principal axes of stress do not coincide with the Cartesian axes. However, in the special case in which *the two families of fibres are mechanically equivalent* the strain energy must be symmetric with respect to interchange of I_4 and I_6 and of I_5 and I_7. Since, for the considered deformation, we have $I_4 = I_6, I_5 = I_7$ it follows that $W_4 = W_6, W_5 = W_7$ and hence, from (96), that $\sigma_{12} = 0$. The principal axes of stress then coincide with the Cartesian axes and $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are just the principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$.

In view of (91) and the fact that I_4, I_5, I_8 depend on λ_1, λ_2 and φ, we may regard the strain energy as a function of λ_1, λ_2 and φ. We write $\hat{W}(\lambda_1, \lambda_2, \varphi)$, but it should be emphasized that, as for the transversely isotropic case and unlike for isotropic materials, $\hat{W}$ is *not* symmetric with respect to interchange of any pair of the stretches. Extending the definition (82) we have

$$\hat{W}(\lambda_1, \lambda_2, \varphi) = \check{W}(I_1, I_2, I_4, I_5, I_6, I_7, I_8), \tag{98}$$

with (91)–(93), and it is straightforward to check that

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{99}$$

which are identical *in form* to equations (51) except that here $\hat{W}$ depends on φ and is not (in general) symmetric in (λ_1, λ_2). These equations describe an *orthotropic* material with the axes of orthotropy coinciding with the Cartesian axes.

Extension and inflation of a thick-walled tube. For the extension and inflation of a thick-walled tube the deformation was examined in Section 1.5.2. Since this deformation is locally a pure homogeneous strain the results discussed for an isotropic material carry over to the considered anisotropic material with the fibre directions $\mathbf{M}$ and $\mathbf{M}'$ locally in the (Θ, Z)-plane symmetrically disposed with respect to the axial direction. The cylindrical polar directions are then the principal directions of strain (and stress) and the strain energy may be written in the form

$$\hat{W}(\lambda, \lambda_z, \varphi), \tag{100}$$

where, as in Section 1.5.2, $\lambda = \lambda_2$ and $\lambda_z = \lambda_3$ respectively are the azimuthal and axial stretches. Furthermore, the formulas (67) and (68) again apply. It is worth emphasizing that they remain valid if the fibre directions depend on the radius, i.e. if φ depends on R, and, in particular, if there are two or more concentric layers with different values of φ. For convenience, we repeat equation (67) here as

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi)\mathrm{d}\lambda, \tag{101}$$

with the arguments of $\hat{W}$ made explicit.

It is a straightforward matter to evaluate the integral in (101) for particular choices of energy function, as was illustrated in the case of isotropy in Section 1.5.2. As shown in Figure 3 the qualitative character of the results is essentially independent of the tube wall thickness and sufficient information concerning the dependence of the pressure-stretch response on the degree of anisotropy can therefore be determined by considering the thin-wall (membrane) approximation of (101). This has the form

$$P = \epsilon\lambda^{-1}\lambda_z^{-1}\frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi), \tag{102}$$

where $\epsilon = H/A$, $H = B - A$ being the wall thickness in the reference configuration, and λ represents any value of the azimuthal stretch through the wall (such values differ only by a term of order ϵ).

For definiteness we consider an energy function that is a natural extension of (69) to the type of anisotropy considered here. This has the form

$$\begin{aligned}\hat{W}(\lambda, \lambda_z, \varphi) = {} & [\mu_1(\varphi)(\lambda^n - 1 - n\ln\lambda) + \mu_2(\varphi)(\lambda_z^n - 1 - n\ln\lambda_z) \\ & + \mu_3(\lambda^{-n}\lambda_z^{-n} - 1 + n\ln(\lambda\lambda_z))]/n,\end{aligned} \tag{103}$$

where the logarithmic terms are needed to ensure that the stresses vanish in the undeformed configuration, μ_3 is a material constant and $\mu_1(\varphi)$ and $\mu_2(\varphi)$ are material parameters dependent of the angle φ. Equation (103) is a special case of a form of energy function used by Ogden and Schulze-Bauer (2000). Note that (69) is recovered by setting $\mu_1 = \mu_2 = \mu_3 = 2\mu/n$.

In respect of (103) equation (102) gives, in dimensionless form,

$$P^* \equiv \lambda_z P/\epsilon\mu_3 = \mu_1^*\lambda^{n-2} - (\mu_1^* - 1)\lambda^{-2} - \lambda^{-n-2}\lambda_z^{-n}, \tag{104}$$

where $\mu_1^* = \mu_1/\mu_3$. Note that (104) is independent of μ_2. The results for isotropy are recovered by setting $\mu_1^* = 1$. The material can be regarded as reinforced (weakened) in the circumferential direction (relative to the radial direction) if $\mu_1^* > 1$ ($\mu_1^* < 1$). Results for $\mu_1^* = 0.5, 1, 2$ are plotted in Figure 5 for comparison, with λ_z set to the value 1.2, as for Figure 3. We recall that for isotropy the inequality $\lambda^2\lambda_z \geq 1$ must hold for inflation following an initial axial stretch. For an anisotropic material this must be replaced by an inequality on λ^n whose lower limit is determined by setting $P = 0$ in (104). This is reflected in the curves in Figure 5, which cut the λ axis at different points. The upper, middle and lower curves in Figure 5 correspond to $\mu_1^* = 2, 1, 0.5$ respectively. For illustrative purposes only the value $n = 10$ has been used for the above calculations.

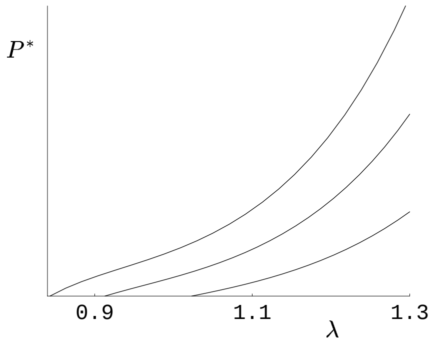

Figure 5. Plot of the dimensionless pressure P^* against the azimuthal stretch λ for fixed λ_z and for values $\mu_1^* = 2, 1, 0.5$ of the anisotropy parameter, corresponding to the upper, middle and lower curves respectively.

The membrane counterpart of (102) for equation (68) has the form

$$F/\pi A^2 \equiv N/\pi A^2 - P\lambda^2 = \epsilon\Big[2\frac{\partial\hat{W}}{\partial\lambda_z}(\lambda,\lambda_z,\varphi) - \lambda\lambda_z^{-1}\frac{\partial\hat{W}}{\partial\lambda}(\lambda,\lambda_z,\varphi)\Big], \qquad (105)$$

where F is interpreted as the *reduced force* on the ends of the tube. With $F = 0$ the equivalent of equations (102) and (105) (in different notation) were solved numerically by Holzapfel and Gasser (2001) in respect of an energy function containing a term exponential in I_4 to obtain plots of the pressure as a function of λ and (separately and equivalently) of λ_z. This reveals some interesting characteristics. For a certain range of values of φ the pressure increases monotonically with λ but for other values it is non-monotonic and λ decreases initially as the pressure increases before ultimately reaching a minimum and then increasing. Similarly for the dependence of P on λ_z. This so-called *inversion effect* is a reflection of the inversion seen in the actual data for arteries in Figure 2(b). The turning points are determined by solution of the equation

$$\lambda^2\frac{\partial^2\hat{W}}{\partial\lambda^2}(\lambda,\lambda_z,\varphi) + \lambda\frac{\partial\hat{W}}{\partial\lambda}(\lambda,\lambda_z,\varphi) - 2\lambda\lambda_z\frac{\partial^2\hat{W}}{\partial\lambda\partial\lambda_z}(\lambda,\lambda_z,\varphi) = 0 \qquad (106)$$

in conjunction with (105) for constant $F^* = F/\epsilon\pi A^2$, as pointed out by Ogden and Schulze-Bauer (2000). Equation (106) is obtained from (102) and (105) by setting $\mathrm{d}\lambda_z/\mathrm{d}P = 0$ at constant F. In order to predict the inversion effect using an energy function of the type (103) it needs to be extended to include two or more terms with different values of n, but we omit details of this here. It should be pointed out that this inversion effect is not a feature specific to anisotropy since it may be found for particular choices of isotropic energy function.

Although the membrane approximation gives a good qualitative picture of the pressure-stretch behaviour it should be used with caution. For example, membrane theory is not able to account for the through-thickness stress distribution in arterial walls or the important influence of residual stresses that are present in arterial wall layers. To account for these influences it is necessary to use a 'thick-wall' model. Both residual stresses and stress distributions through the wall thickness will be discussed in Section 4.

3 Convexity and Material Stability

The concepts of convexity and material stability are closely related and have important roles to play in the construction of constitutive laws for soft biological tissues and for the analysis of boundary-value problems. In this section we examine some connections between these notions and, in particular, between the convexity of contours of constant strain energy and local material stability. One point to emphasize is that these notions depend on the choice of strain or deformation measure and, for purposes of illustration, we begin, in the following section, by considering constant energy contours for two specific forms of energy function, one isotropic and one anisotropic.

3.1 Contours of constant energy

In the notation $\check{W}$ that we have used here to represent the strain-energy function for an incompressible material when regarded as a function of the invariants, the (isotropic) strain-energy function introduced by Delfino et al. (1997) is given by

$$\check{W}(I_1) = \frac{a}{b}\left\{\exp\left[\frac{b}{2}(I_1 - 3)\right]\right\}, \tag{107}$$

where a and b are positive material constants and I_1 is the invariant defined by $(8)_1$ or, after use of the incompressibility condition, by $(91)_1$. Contours of constant of $\check{W}(I_1)$, when plotted in the (λ_1, λ_2) plane are the same as contours of constant I_1, albeit with a different constant. Such contours are plotted in Figure 6 for several constant values of I_1. The contours are clearly convex, and, as we shall see in Section 3.4, this convexity is a consequence of material stability under dead-loading conditions. If the corresponding contours are plotted, instead, in the plane of constant (e_1, e_2), where

$$e_1 = \frac{1}{2}(\lambda_1^2 - 1), \quad e_2 = \frac{1}{2}(\lambda_2^2 - 1) \tag{108}$$

are the principal components of the Green strain tensor (9) then this convexity is also evident, as shown by Holzapfel et al. (2000).

This situation contrasts with that for the (anisotropic) Fung-type strain-energy function (Fung et al., 1979) defined, in notation adapted to the present context, by

$$\check{W}(Q) = \frac{\mu}{2}\Big[\exp(Q) - 1\Big], \tag{109}$$

where μ is a material constant and Q is specified as

$$Q = ae_1^2 + 2be_1e_2 + ce_2^2, \tag{110}$$

where a, b, c are dimensionless material constants and (e_1, e_2) are as given in (108). Contours of constant $\check{W}(Q)$ are also those of constant Q. In the (e_1, e_2) plane these contours are convex

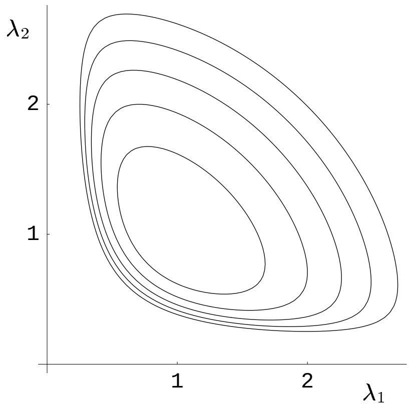

Figure 6. Plot of contours of constant I_1 in the (λ_1, λ_2) plane.

provided $a > 0, c > 0, ac > b^2$, and an example of the plots is shown in Figure 7(a) (necessarily, $e_1 > -1/2$ and $e_2 > -1/2$, corresponding to $\lambda_1 > 0$ and $\lambda_2 > 0$). However, it does not in general follow that the contours of constant Q are convex in (λ_1, λ_2) space for the same values of a, b, c. This is illustrated in Figure 7(b). The lack of convexity implies, as we shall show in a more general setting in Section 3.3, that a material with the given values of a, b, c may become unstable under dead loading conditions.

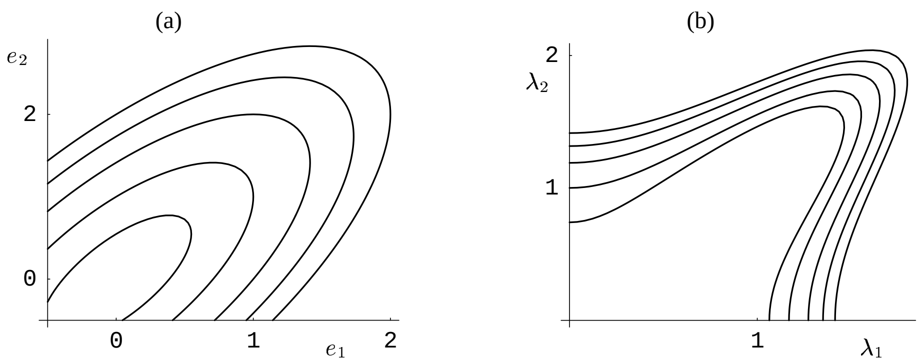

Figure 7. Plot of contours of constant Q in the (e_1, e_2) plane (a) and the (λ_1, λ_2) plane (b).

The point that needs to be made here is that special care should be taken in developing forms of strain-energy function for the modelling of soft tissue (and, of course, for other materials). Convexity in one strain measure (such as Green strain) does not necessarily guarantee material stability. Equally, lack of convexity may not preclude material stability, but failure of convexity may have undesirable consequences for the development of numerical schemes for the solution of (initial-) boundary-value problems. We shall elaborate on these points in the following sections.

3.2 The notion of stability for simple tension

We recall from (54) that for simple tension the Biot stress t is given in terms of the stretch λ by

$$t = \tilde{W}'(\lambda). \tag{111}$$

With reference to Figure 1 we see that t is a monotonic increasing function of λ for the two materials considered there. Thus,

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \tilde{W}''(\lambda) > 0. \tag{112}$$

This is a mathematical statement of the fact that $\tilde{W}$ is (locally) *strictly convex* as a function of λ. If strict inequality is replaced by $\geq$ in (112) then $\tilde{W}$ is (locally) *convex*. If (112) holds for all $\lambda > 0$ then $\tilde{W}$ is (globally) *strictly convex* as a function of λ, and the curve of $\tilde{W}$ as a function of λ is bowl shaped upwards and any straight line joining two points of the curve lies above the curve. The extension of this idea to $\hat{W}(\lambda_1, \lambda_2)$ will be considered in Section 3.4 but first we examine how the above considerations change when the stress and strain variables change.

Let $e = (\lambda^2 - 1)/2$ and let $\tau = \bar{W}'(e)$ be the corresponding component of the second-Piola Kirchhoff stress (the specialization of $(27)_2$), where $\bar{W}(e) \equiv \tilde{W}(\lambda)$. Then $\tau = \lambda^{-1}t$ and

$$\tilde{W}''(\lambda) = \lambda^2 \bar{W}''(e) + \bar{W}'(e) = \lambda^2 \frac{\mathrm{d}\tau}{\mathrm{d}e} + \tau. \tag{113}$$

Thus, in tension, for example, the curve of τ as a function of e can have the non-convex shape shown in Figure 8 without violating the inequality (112). This is another manifestation of the discussion in Section 3.1.

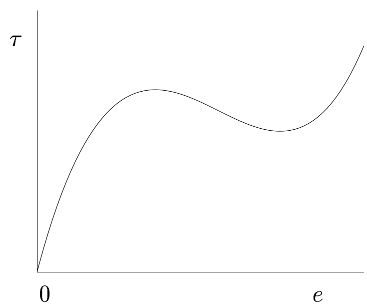

Figure 8. Plot of τ as a function of e.

For homogeneous simple tension under dead load equation (111) may be obtained in an elementary way by considering the variation with respect to λ of the 'energy function' $\tilde{E}(\lambda)$

defined by

$$\tilde{E}(\lambda) = \tilde{W}(\lambda) - t(\lambda - 1), \tag{114}$$

this being the excess of the stored energy over the work done by the load t in extending the specimen to a stretch λ. The first variation $\delta\tilde{E}$ of (114) with respect to λ is simply

$$\delta\tilde{E}(\lambda) = (\tilde{W}'(\lambda) - t)\delta\lambda, \tag{115}$$

where $\delta\lambda$ is the variation in λ. Clearly, $\tilde{E}(\lambda)$ is stationary if and only if (111) holds. The second variation of $\tilde{E}(\lambda)$ is then easily seen to be $\delta^2\tilde{E} = \tilde{W}''(\lambda)(\delta\lambda)^2$, and stability (under dead load) for this one-dimensional situation therefore requires that (112) holds. In this one-dimensional situation convexity of $\tilde{W}$ and (material) stability are equivalent. This is not in general the case in two or three dimensions, as we shall see in Section 3.4. Incidentally, the term -1 in (114) can be omitted without affecting the results since it corresponds to the addition of a constant to $\tilde{E}(\lambda)$.

It is worth noting in passing how the above arguments are modified if the loading is not 'dead'. Suppose that the loading is instead of 'follower' type. Specifically, suppose that it corresponds to the one-dimensional (tension) specialization of the pressure loading condition (41). Then we may write $t = f\lambda^{-1}$, where $f > 0$ is independent of λ. The counterpart of (114) in this case is easily shown to be

$$\tilde{E}(\lambda) = \tilde{W}(\lambda) - f\ln\lambda, \tag{116}$$

and the stability inequality (112) is modified to

$$\lambda\tilde{W}''(\lambda) + \tilde{W}'(\lambda) > 0, \tag{117}$$

which, for tension, is less restrictive than (112).

3.3 Stability in pure homogeneous strain

We next consider an isotropic strain-energy function expressed in terms of the principal stretches. Then, analogously to (114), we consider the total energy

$$E(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_2, \lambda_3) - \sum_{i=1}^{3} t_i\lambda_i \tag{118}$$

in respect of an unconstrained material. For dead loading, stationarity of (118) with respect to the (independent) stretches leads to

$$t_i = \frac{\partial W}{\partial\lambda_i} \quad i \in \{1, 2, 3\}. \tag{119}$$

For E to be a (local) minimum in respect of (119), i.e. for the second variation of E in the stretch variations $\delta\lambda_1, \delta\lambda_2, \delta\lambda_3$ to be positive, we must have

$$\delta^2 E \equiv \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}\delta\lambda_i\delta\lambda_j > 0, \tag{120}$$

and hence the Hessian matrix (W_{ij}) is positive definite, where a subscript i indicates partial differentiation with respect to the stretch λ_i.

If the stretch variations satisfy the incompressibility constraint

$$\sum_{i=1}^{3} \delta\lambda_i/\lambda_i = 0, \tag{121}$$

obtained by taking the variation of $\ln(\lambda_1\lambda_2\lambda_3) = 0$, (120) can be expressed as

$$\begin{aligned}&(\lambda_1^2 W_{11} - 2\lambda_1\lambda_3 W_{13} + \lambda_3^2 W_{33})(\delta\lambda_1/\lambda_1)^2\\ &+ (\lambda_2^2 W_{11} - 2\lambda_2\lambda_3 W_{13} + \lambda_3^2 W_{33})(\delta\lambda_2/\lambda_2)^2\\ &+ 2(\lambda_1\lambda_2 W_{12} - \lambda_1\lambda_3 W_{13} - \lambda_2\lambda_3 W_{23} + \lambda_3^2 W_{33})(\delta\lambda_1\delta\lambda_2/\lambda_1\lambda_2) > 0.\end{aligned} \tag{122}$$

Further, by taking $\lambda_1, \lambda_2, \lambda_3$ to correspond to an isochoric deformation and adopting the notation defined in (50), (122) may be simplified to give

$$\begin{aligned}&(\lambda_1^2 \hat{W}_{11} - 2\sigma_3)(\delta\lambda_1/\lambda_1)^2 + (\lambda_2^2 \hat{W}_{22} - 2\sigma_3)(\delta\lambda_2/\lambda_2)^2\\ &\quad + (\lambda_1\lambda_2 \hat{W}_{12} - \sigma_3)(\delta\lambda_1\delta\lambda_2/\lambda_1\lambda_2) > 0,\end{aligned} \tag{123}$$

where $\sigma_3 = \lambda_3 W_3$. In fact, it may be shown that (123) is the appropriate modification of (120) for an incompressible material provided σ_3 is set as $\sigma_3 = \lambda_3 W_3 - p$, where p is the arbitrary hydrostatic pressure associated with the incompressibility constraint. For a direct derivation of (123) for an incompressible material we refer to Ogden (1997).

Thus, the inequality (123) is the dead-loading stability requirement for a pure homogeneous strain subject to the restriction that departure from a state of pure homogeneous strain is not allowed. Further restrictions are needed if deformation variations that allow the orientation of the principal axes of strain to change are admitted. This will be discussed in Section 3.5.

For the special case in which $\sigma_3 = 0$, the inequality (123) reduces simply to the requirement

$$\text{Hessian matrix} \quad \begin{pmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{12} & \hat{W}_{22} \end{pmatrix} \quad \text{is positive definite.} \tag{124}$$

This condition is appropriate, for example, for a thin sheet of material under plane stress and biaxial tension.

The inequalities (123) and (124) apply not just for isotropic materials but also for anisotropic materials constructed on the basis of (98) under the restriction to pure homogeneous strain. If $\sigma_3 \neq 0$ then (123) puts restrictions on the range of values of σ_3 that can be supported in a given deformed configuration.

The distinction between results for different strain measures can now be brought out very clearly, and we illustrate this, as in the case of simple tension, in respect of the Green strain measure. Let $(\bar{W}_{ij})$ denote the Hessian matrix of W with respect to the principal Green strains (e_1, e_2) and let $\delta e_1, \delta e_2$ be the variations in (e_1, e_2) associated with $\delta\lambda_1, \delta\lambda_2$. It follows that

$$\sum_{i=1}^{2} \hat{W}_{ij}\delta\lambda_i\delta\lambda_j = \sum_{i=1}^{2} \bar{W}_{ij}\delta e_i\delta e_j + \bar{W}_1(\delta\lambda_1)^2 + \bar{W}_2(\delta\lambda_2)^2. \tag{125}$$

Thus, we emphasize that local (strict) convexity of $\bar{W}(e_1, e_2)$ does not in general imply material stability under dead loading, i.e. local (strict) convexity of $\hat{W}(\lambda_1, \lambda_2)$ in the present (restricted) context since the principal stresses $\bar{W}_1$ and $\bar{W}_2$ may be negative. In the special case of a thin sheet under tension, however, the implication does follow. On the other hand, failure of local (strict) convexity of $\bar{W}(e_1, e_2)$ does not necessarily mean failure of stability since the stresses $\bar{W}_1$ and $\bar{W}_2$ may be positive.

3.4 Convexity connections

Consider the contour in (λ_1, λ_2) space defined by

$$\hat{W}(\lambda_1, \lambda_2) = c, \tag{126}$$

where c is a positive constant. On differentiation of (126) we obtain

$$\hat{W}_1 \lambda_1' + \hat{W}_2 \lambda_2' = 0, \tag{127}$$

where (λ_1', λ_2') is the unit tangent vector to the contour (126), the prime indicating differentiation with respect to the arclength parameter.

The curvature of (126), denoted κ, is given by the standard formula

$$\kappa = \lambda_1' \lambda_2'' - \lambda_1'' \lambda_2'. \tag{128}$$

On using both (127) and its derivative with respect to the arclength parameter in (128) we obtain, after some rearrangement,

$$\kappa = [\hat{W}_{11}(\lambda_1')^2 + 2\hat{W}_{12}\lambda_1'\lambda_2' + \hat{W}_{22}(\lambda_2')^2]\lambda_2'/\hat{W}_1. \tag{129}$$

This shows the connection between κ and the quadratic form based on the Hessian $(\hat{W}_{ij})$. Note, however, that in (129) the vector (λ_1', λ_2') is not arbitrary but is restricted to the tangential direction to (126). Note also that the sign of κ is dependent on the sense of description of the contour. Here, this description is taken in the anticlockwise sense so that, for example, the curvature of a circle of radius a with centre at the origin is $+1/a$.

If the contour (126) is convex then $\kappa > 0$ and it may be deduced from (129) that

$$\hat{W}_{11}(\lambda_1')^2 + 2\hat{W}_{12}\lambda_1'\lambda_2' + \hat{W}_{22}(\lambda_2')^2 > 0 \tag{130}$$

since, as we shall show below, the inequality $\lambda_2'/\hat{W}_1 > 0$ may be adopted independently of (130). We emphasize that in (130) (λ_1', λ_2') is restricted as indicated above and hence it does *not* follow that the Hessian $(\hat{W}_{ij})$ is positive definite. On the other hand, if $(\hat{W}_{ij})$ is positive definite then (130) follows, $\kappa > 0$ and the contour is convex. Thus, if the contour is not convex the stability inequality

$$\sum_{i,j=1}^{2} \hat{W}_{ij}\delta\lambda_i\delta\lambda_j > 0, \quad \delta\lambda_i \not\equiv 0, \tag{131}$$

obtained from (123), cannot hold. A check on the convexity of the contour (126) therefore provides a quick test of the suitability of a chosen form of strain-energy function.

To illustrate this we consider the neo-Hookean strain-energy function

$$\hat{W}(\lambda_1, \lambda_2) = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3), \tag{132}$$

where $\mu > 0$ is the shear modulus of the material. This is a special case of (69), corresponding to $n = 2$. Then,

$$\hat{W}_1 = \mu(\lambda_1 - \lambda_1^{-3}\lambda_2^{-2}), \quad \hat{W}_2 = \mu(\lambda_2 - \lambda_1^{-2}\lambda_2^{-3}), \tag{133}$$

from which it follows that

$$\begin{aligned} \hat{W}_1 \geq 0 \ (\leq 0) \quad \text{when} \quad \lambda_1^2\lambda_2 \geq 1 \ (\leq 1), \\ \hat{W}_2 \geq 0 \ (\leq 0) \quad \text{when} \quad \lambda_1\lambda_2^2 \geq 1 \ (\leq 1). \end{aligned} \tag{134}$$

It is easy to show that $(\hat{W}_{ij})$ is positive definite and that, with reference to (127), $\lambda_2'/\hat{W}_1 > 0$ and $\lambda_1'/\hat{W}_2 < 0$. In Section 3.5.1 we shall conclude that the latter inequalities are valid more generally, not just for (132). Figure 9 shows a typical contour of constant $\hat{W}(\lambda_1, \lambda_2)$, as in Figure 6, together with the curves $\lambda_1^2\lambda_2 = 1$ and $\lambda_1\lambda_2^2 = 1$, corresponding to $\hat{W}_1 = 0$ and $\hat{W}_2 = 0$ respectively. Note that these curves cut the contour where the tangent is either horizontal or vertical and that the contour is symmetric about the line $\lambda_1 = \lambda_2$.

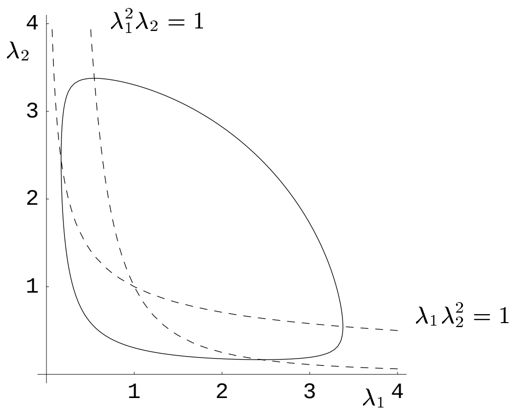

Figure 9. Contour plot for the neo-Hookean strain energy (132) and the (dashed) curves $\lambda_1^2\lambda_2 = 1$ and $\lambda_1\lambda_2^2 = 1$ in the (λ_1, λ_2) plane.

For comparison with Figure 9 we show, in Figure 10, a corresponding picture for a typical (convex) anisotropic strain energy based on (103). The lack of symmetry should be noted in this case.

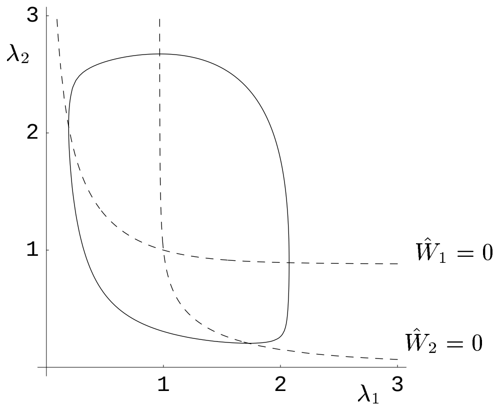

Figure 10. Contour plot for a typical member of the class of anisotropic strain-energy functions (103) and the (dashed) curves $\hat{W}_1 = 0$ and $\hat{W}_2 = 0$ in the (λ_1, λ_2) plane.

3.5 Stability under dead loading: the general case

In deriving the inequality (120) we have discounted the possibility that the orientation of the principal axes of strain can change during. If this is to be allowed for then (118) must be replaced by either

$$E(\mathbf{U}) = W(\mathbf{U}) - \mathrm{tr}(\mathbf{TU}) \tag{135}$$

or

$$E(\mathbf{F}) = W(\mathbf{F}) - \mathrm{tr}(\mathbf{SF}), \tag{136}$$

where tr denotes the trace of a second order tensor. The first variations of these with respect to **U** and **F** respectively lead to equations $(26)_1$ and (18). Note that the difference between these lies in the fact that (136) allows for rigid rotations through **R** and its variation. The conditions for stability are respectively (at fixed **T**)

$$\mathrm{tr}\,\{(\boldsymbol{\mathcal{L}}\delta\mathbf{U})\delta\mathbf{U}\} > 0 \tag{137}$$

for all $\delta\mathbf{U} \neq \mathbf{0}$ and (at fixed **S**)

$$\mathrm{tr}\,\{(\boldsymbol{\mathcal{A}}\delta\mathbf{F})\delta\mathbf{F}\} > 0 \tag{138}$$

for all $\delta\mathbf{F} \neq \mathbf{0}$, where the fourth-order tensors $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{A}}$ are defined by

$$\boldsymbol{\mathcal{L}} = \frac{\partial^2 W}{\partial \mathbf{U}^2}, \quad \boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F}^2}. \tag{139}$$

The component form of $\boldsymbol{\mathcal{A}}$ is given by (44) and that for $\boldsymbol{\mathcal{L}}$ is defined analogously.

It is well known that the inequality (138) *cannot* hold in all configurations, and hence that $\boldsymbol{\mathcal{A}}$ is singular in certain configurations when regarded as a linear mapping on the (nine-dimensional) space of variations $\delta\mathbf{F}$. We refer to, for example, Ogden (1991, 1997, 2000) for detailed analysis of the singularities of $\boldsymbol{\mathcal{A}}$ and their implications for bifurcation in the dead-load problem.

Modification of these results in the incompressible case requires use of the variations of the incompressibility condition $\det\mathbf{F} = \det\mathbf{U} = 1$, in the form

$$\mathrm{tr}\,(\mathbf{U}^{-1}\delta\mathbf{U}) = \mathrm{tr}\,(\mathbf{F}^{-1}\delta\mathbf{F}) = 0, \tag{140}$$

but we do not give details for the general case here.

Specialization to isotropy For the important special case of an isotropic material it is useful to give explicit expressions for the components of $\boldsymbol{\mathcal{A}}$. The (non-zero) components of $\boldsymbol{\mathcal{A}}$ referred to the principal axes $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are given by

$$\mathcal{A}_{iijj} = W_{ij}, \tag{141}$$

$$\mathcal{A}_{ijij} - \mathcal{A}_{ijji} = \frac{W_i + W_j}{\lambda_i + \lambda_j} \quad i \neq j, \tag{142}$$

$$\mathcal{A}_{ijij} + \mathcal{A}_{ijji} = \frac{W_i - W_j}{\lambda_i - \lambda_j} \quad i \neq j,\ \lambda_i \neq \lambda_j, \tag{143}$$

$$\mathcal{A}_{ijij} + \mathcal{A}_{ijji} = W_{ii} - W_{ij} \quad i \neq j,\ \lambda_i = \lambda_j, \tag{144}$$

where $W_i = \partial W/\partial\lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$, $i, j \in \{1, 2, 3\}$, and no summation is implied by the repetition of indices. For details of the derivation of these components we refer to Ogden (1997). Equations (141)–(144) apply for both compressible and incompressible materials subject, in the latter case, to the constraint (10).

In the case of an isotropic material the local stability inequality (138) can be given explicitly in terms of the derivatives of the strain-energy function with respect to the stretches. This leads to

$$\begin{aligned}\mathrm{tr}\,\{(\boldsymbol{\mathcal{A}}\delta\mathbf{F})\delta\mathbf{F}\} = \sum_{i,j=1}^{3} W_{ij}\delta\lambda_i\delta\lambda_j + \sum_{i\neq j}(W_i - W_j)(\lambda_i - \lambda_j)\Big(\mathit{\Omega}^L_{ij} + \frac{1}{2}\mathit{\Omega}^R_{ij}\Big)^2 \\ + \frac{1}{4}\sum_{i\neq j}(W_i + W_j)(\lambda_i + \lambda_j)\big(\mathit{\Omega}^R_{ij}\big)^2 > 0,\end{aligned} \tag{145}$$

where $\mathit{\Omega}^L_{ij} = \mathbf{u}^{(i)}\cdot\delta\mathbf{u}^{(j)}$ are the components of the (antisymmetric) tensor $\boldsymbol{\mathit{\Omega}}^L$ that defines the variation $\delta\mathbf{u}^{(i)}$ through $\delta\mathbf{u}^{(i)} = \boldsymbol{\mathit{\Omega}}^L\mathbf{u}^{(i)}$, and $\boldsymbol{\mathit{\Omega}}^R = \mathbf{R}^T\delta\mathbf{R}$ is also antisymmetric, $\delta\mathbf{R}$ being the variation in **R**.

Since $\delta\lambda_i$, $\mathit{\Omega}^L_{ij}$, $\mathit{\Omega}^R_{ij}$ are independent, necessary and sufficient conditions for (145) to hold are therefore

$$\text{Hessian matrix } (W_{ij}) \text{ is positive definite,} \tag{146}$$

$$\frac{W_i - W_j}{\lambda_i - \lambda_j} > 0 \qquad i \neq j, \tag{147}$$

$$W_i + W_j > 0 \qquad i \neq j \tag{148}$$

jointly for $i, j \in \{1, 2, 3\}$. Note that when $\lambda_i = \lambda_j$, $i \neq j$ (147) reduces to $W_{ii} - W_{ij} > 0$.

The inequality (146) is just that arising from (120). In respect of (137) only (146) and (147) are required. Thus, allowing for the change in orientation of the Lagrangian principal axes imposes the additional inequalities (147) over and above (120). These are obtained by setting $\delta\mathbf{R} = \mathbf{0}$ in (145). It is worth noting, however, that it can be shown that (147) follow from (146) if the region of $(\lambda_1, \lambda_2, \lambda_3)$ space on which (146) holds is convex. The additional inequalities (148) arise solely from geometrical considerations associated with the rigid rotation **R**.

For the case of plane stress of a thin sheet we may take $W_3 = 0$ and it then follows from (148) that $W_1 > 0$, $W_2 > 0$ and then from (147) that $\lambda_1 > \lambda_3$ and $\lambda_1 > \lambda_3$.

For an incompressible material the counterparts of (146)–(148) are

$$\text{matrix} \begin{bmatrix} \lambda_1^2 \hat{W}_{11} - 2\sigma_3 & \lambda_1\lambda_2\hat{W}_{12} - \sigma_3 \\ \lambda_1\lambda_2\hat{W}_{12} - \sigma_3 & \lambda_2^2\hat{W}_{22} - 2\sigma_3 \end{bmatrix} \text{ is positive definite,} \tag{149}$$

$$\frac{t_i - t_j}{\lambda_i - \lambda_j} > 0 \quad i \neq j, \tag{150}$$

$$t_i + t_j > 0 \quad i \neq j. \tag{151}$$

Note that p occurs in (149)–(151) implicitly through σ_3 and t_i, $i \in \{1, 2, 3\}$.

For the case of plane stress with $t_3 = 0$ we may deduce that $\hat{W}_1 > 0$, $\hat{W}_2 > 0$ and $\lambda_1^2\lambda_2 > 1$, $\lambda_1\lambda_2^2 > 1$ as was discussed in Section 3.4 in respect of the neo-Hookean strain energy. The inequalities (150) on their own are entirely consistent with the factor $\lambda_2'/\hat{W}_1$ in (129) being positive since, by (127), $\hat{W}_1$ and λ_2' change sign together where the curve $\lambda_1^2\lambda_2 = 1$ crosses the contour (126). Indeed, it can be shown that this factor is positive even for non-convex contours, an example of which is given by Ogden and Roxburgh (1999).

3.6 Elastic moduli in the classical limit

In the classical theory of elasticity, corresponding to the situation in which there is no underlying deformation or stress, the components of $\boldsymbol{\mathcal{A}}$ for an isotropic material can be written compactly in the form

$$\mathcal{A}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{152}$$

where λ and μ are the classical Lamé moduli of elasticity and δ_{ij} is the Kronecker delta. The values of W_{ij} when $\lambda_i = 1$ for $i \in \{1, 2, 3\}$ are simply $W_{ii} = \lambda + 2\mu$, $W_{ij} = \lambda$, $i \neq j$. Also, we take $W_i = 0$ when $\lambda_j = 1$ for $i, j \in \{1, 2, 3\}$ so that the configuration $\mathcal{B}_r$ is a stress free (*natural*) configuration.

The counterpart of (152) for an incompressible material is

$$\mathcal{A}_{iiii} = \mathcal{A}_{ijij} = \mu, \quad \mathcal{A}_{iijj} = \mathcal{A}_{ijji} = 0 \quad i \neq j, \tag{153}$$

and $W_{ii} = W_i = \mu$, $W_{ij} = 0$, where μ is the *shear modulus* in $\mathcal{B}_r$. The expressions in (153) are not uniquely defined because they depend on the point at which $\lambda_1\lambda_2\lambda_3$ is set to unity in the differentiations in (141)–(144) prior to setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$. In terms of the strain-energy function $\hat{W}$ defined in (50) the restrictions required in $\mathcal{B}_r$ may be written

$$\hat{W}(1,1) = 0, \quad \hat{W}_\alpha(1,1) = 0, \quad \hat{W}_{12}(1,1) = 2\mu, \quad \hat{W}_{\alpha\alpha}(1,1) = 4\mu, \tag{154}$$

where the index α is 1 or 2.

Finally in this section we note that for an anisotropic material more material constants are needed to characterize the material properties in the classical limit than are given for the isotropic case. For example, for the strain energy function defined by (98), $\hat{W}_{11}(1,1,\varphi)$, $\hat{W}_{22}(1,1,\varphi)$ and $\hat{W}_{12}(1,1,\varphi)$ are in general independent constants, as can be checked from the example (103).

4 Residual Stresses and Arterial Wall Mechanics

When a length of artery is excised from a body it contracts in length. The resulting configuration of the specimen is referred to as an *unloaded* configuration, i.e. it is subject to no axial load nor to any tractions on its inner and outer surfaces. However, in this configuration there remain stresses through the artery wall. This is demonstrated by cutting radially a short length of artery in the form of a ring. The ring springs open to form an open sector (Vaishnav and Vossoughi, 1983; see, also, Fung, 1993). In general even this open sector is not stress free since the opening angles of circumferentially separated layers are different (Vossoughi et al., 1993, and Greenwald et al., 1997).

In most analyses, however, the opened-up sector is assumed, for simplicity, to be stress free in order to facilitate calculation of the (residual) stress required to re-form the intact ring (the unloaded configuration) as in, for example, Fung and Liu (1989), Delfino et al. (1997) and Holzapfel et al. (2000). It is normally assumed that the ring is a circular annulus and that the opened-up sector is also circular and that the deformation required to re-form the ring depends only on the radius. It should be noted, however, as we show in Section 4.4, that these kinematic assumptions, when coupled with the equilibrium equations and boundary conditions, necessarily require the opened-up sector to be stress free. Any assumptions that are less simple than these would almost certainly require a purely numerical treatment.

The residual stresses have an influence on the overall behaviour of an artery under extension and internal pressure and, more significantly, on the stress and strain distributions through the arterial wall. It has been suggested in the literature that in the physiological state a healthy artery has an essentially constant circumferential stress in each layer of its wall. This can only be the case if there is residual stress present (see the discussion in Rachev and Hayashi, 1999, for example). It has also been conjectured (Takamizawa and Hayashi, 1987) that the strain distribution is constant through the wall. Some consequences of the assumptions of uniform circumferential stress and/or uniform strain have been examined by Ogden and Schulze-Bauer (2000).

The opening angle experiment gives only a very rough estimate of the residual stress, and a detailed understanding of the mechanical influence of residual stress therefore remains to be developed. Influences that need to be accounted for are, *inter alia*, growth, remodelling and adaptation since these clearly generate residual stresses. Analysis of such effects is at an early stage of development and much more needs to be done in this area. Recent contributions to these issues are, for example, Rodriguez et al. (1994), Rachev (1997, 2000), Rachev et al. (1998) and Taber and Humphrey (2001). See, also, the chapter by Rachev (2001) in this volume.

Some fundamental aspects of the influence of residual stress on the constitutive law of a nonlinearly elastic solid have been examined in a series of papers by Hoger and co-workers (see, for example, Hoger, 1985, 1993, and Johnson and Hoger, 1998). We begin the discussion of residual stress, in Section 4.1, by examining briefly some of these basic theoretical issues.

4.1 Elastic response in the presence of residual stress

In Section 1.3 we imposed the restrictions (22) on the strain-energy function. Now we suppose that the reference configuration $\mathcal{B}_r$ is not stress free, i.e. there is a residual stress in $\mathcal{B}_r$ and $(22)_2$ no longer holds. Let $\boldsymbol{\sigma}^{(r)}$ denote the residual (Cauchy) stress in $\mathcal{B}_r$. In general, this is not obtained from a strain energy, and we may take the strain-energy function W to be measured from $\mathcal{B}_r$ and to satisfy $(22)_1$. Since there is no deformation in $\mathcal{B}_r$ there is no distinction between the Cauchy stress and the nominal stress $\mathbf{S}^{(r)}$ there. The residual stress must satisfy the equilibrium equation

$$\mathrm{Div}\,\mathbf{S}^{(r)} = \mathbf{0} \quad \text{in } \mathcal{B}_r, \tag{155}$$

and, since the boundary is load free, the boundary conditions

$$\mathbf{S}^{(r)T}\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_r. \tag{156}$$

Since

$$\mathrm{Div}\left(\mathbf{S}^{(r)} \otimes \mathbf{X}\right) = \left(\mathrm{Div}\,\mathbf{S}^{(r)}\right) \otimes \mathbf{X} + \mathbf{S}^{(r)}, \tag{157}$$

it follows from (155), (156) and by application of the divergence theorem that

$$\int_{\mathcal{B}_r} \mathbf{S}^{(r)}\mathrm{d}V = \mathbf{0}. \tag{158}$$

An immediate consequence of (158) is that residual stress cannot be uniform — *the residual stress distribution is necessarily inhomogeneous* and is therefore geometry dependent. A further consequence is that the material response of a residually stress body relative to the residually stressed configuration, and hence the constitutive law, is geometry dependent and inhomogeneous.

Residual stress places restrictions on the material symmetry in $\mathcal{B}_r$ and, in view of the above remarks, the material symmetry may therefore vary from point to point within the considered material body. The constitutive laws resulting from these restrictions are, in general, very complicated, and we shall not consider this issue here. We shall, however, examine the simpler issue of what restrictions are imposed on the residual stress by specific material symmetries.

We restrict attention to symmetry groups that consist of elements of the proper orthogonal group. Suppose that $\mathbf{Q}$ is a rotation tensor belonging to a symmetry group relative to $\mathcal{B}_r$. Then, by combining the stress-deformation relation (18) with the objectivity requirement (24) and the material symmetry requirement (28) suitably specialized, we obtain

$$\mathbf{H}(\mathbf{QF}) = \mathbf{H}(\mathbf{F})\mathbf{Q}^T, \tag{159}$$

for all proper orthogonal $\mathbf{Q}$, and

$$\mathbf{H}(\mathbf{FQ}) = \mathbf{Q}^T\mathbf{H}(\mathbf{F}), \tag{160}$$

for all members $\mathbf{Q}$ of the symmetry group. By setting $\mathbf{F} = \mathbf{I}$ and $\mathbf{S}^{(r)} = \mathbf{H}(\mathbf{I})$ and using (159) and (160), we obtain

$$\mathbf{S}^{(r)}\mathbf{Q} = \mathbf{Q}\mathbf{S}^{(r)}, \tag{161}$$

or, equivalently,

$$\mathbf{Q}\boldsymbol{\sigma}^{(r)}\mathbf{Q}^T = \boldsymbol{\sigma}^{(r)}. \tag{162}$$

Equations (161) and (162) must hold for every member of the symmetry group and hence restrictions are imposed on the form of $\mathbf{S}^{(r)}$. More details can be found in, for example, Coleman and Noll (1964) and Hoger (1985).

We now consider three specific material symmetries in order to determine the nature of these restrictions.

Isotropy. For *isotropy* equation (162) must hold for *all* rotations $\mathbf{Q}$. This implies that the residual stress has the form $\boldsymbol{\sigma}^{(r)} = \sigma^{(r)}\mathbf{I}$, where $\sigma^{(r)}$ is a scalar. The equilibrium equation (155) reduces to $\mathrm{Grad}\,\sigma^{(r)} = \mathbf{0}$, so that $\sigma^{(r)}$ is constant. Application of the boundary condition (156) then shows that $\sigma^{(r)} \equiv 0$.

Thus, *residual stress cannot be supported by an isotropic body* whatever the geometry of the body. This is an important result in the context of soft tissue mechanics since it reinforces the need to consider soft tissues as anisotropic materials.

Transverse isotropy. If the material response is *transversely isotropic* relative to $\mathcal{B}_r$ then there is a preferred direction, defined by a unit vector, denoted $\mathbf{k}$, which will in general depend on position in the material. The symmetry group consists of all rotations $\mathbf{Q}$ that preserve $\mathbf{k}$ together with a rotation that reverses $\mathbf{k}$. It may be shown, by following the procedure outlined for the isotropic case, that $\boldsymbol{\sigma}^{(r)}$ must have two equal principal values and be expressible in the form

$$\boldsymbol{\sigma}^{(r)} = \sigma_1^{(r)}(\mathbf{I} - \mathbf{k} \otimes \mathbf{k}) + \sigma_3^{(r)}\mathbf{k} \otimes \mathbf{k}, \tag{163}$$

where $\sigma_1^{(r)} = \sigma_2^{(r)}$ and $\sigma_3^{(r)}$ are the principal values, in general dependent on position. We refer to Hoger (1993) for detailed discussion of (163) and some of its consequences.

Orthotropy. In the case of *orthotropy* the material symmetry identifies three mutually orthogonal directions, here specified by the unit vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. The symmetry group consists of rotations through π about each $\mathbf{k}_i,\ i \in \{1, 2, 3\}$ together with reversal of each $\mathbf{k}_i$. The resulting form of $\boldsymbol{\sigma}^{(r)}$, obtained using (162), is

$$\boldsymbol{\sigma}^{(r)} = \sigma_1^{(r)}\mathbf{k}_1 \otimes \mathbf{k}_1 + \sigma_2^{(r)}\mathbf{k}_2 \otimes \mathbf{k}_2 + \sigma_3^{(r)}\mathbf{k}_3 \otimes \mathbf{k}_3, \tag{164}$$

the principal values of $\boldsymbol{\sigma}^{(r)}$ being distinct and associated with principal directions $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. In general, $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\sigma_1^{(r)}, \sigma_2^{(r)}, \sigma_3^{(r)}$ depend on position. Again, for further discussion and examination of some special cases, we refer to Hoger (1985).

An important special case is that in which one of the principal directions, $\mathbf{k}_3$ say is independent of position. It follows on substitution of (164) into the equilibrium equation (155) that $\sigma_3^{(r)}$ is independent of the Cartesian coordinate associated with $\mathbf{k}_3$. If we identify this direction with the axis of a right circular cylindrical tube then application of the boundary condition (156) on the ends of the tube leads to $\sigma_3^{(r)} \equiv 0$. In terms of the polar coordinates (R, Θ) defined in (57) the remaining components of the equilibrium equation are

$$\frac{\partial \sigma_{RR}^{(r)}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\Theta}^{(r)}}{\partial \Theta} + \frac{\sigma_{RR}^{(r)} - \sigma_{\Theta\Theta}^{(r)}}{R} = 0, \tag{165}$$

$$\frac{\partial \sigma_{R\Theta}^{(r)}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\Theta\Theta}^{(r)}}{\partial \Theta} + 2\frac{\sigma_{R\Theta}^{(r)}}{R} = 0, \tag{166}$$

where $\sigma^{(r)}_{RR}, \sigma^{(r)}_{R\Theta}, \sigma^{(r)}_{\Theta\Theta}$ are the components of $\boldsymbol{\sigma}^{(r)}$.

If there is no dependence on Θ then it follows from (166) and the boundary conditions on the cylindrical surfaces that $\sigma^{(r)}_{R\Theta} \equiv 0$ and hence that $\mathbf{k}_1$ and $\mathbf{k}_2$ coincide with the polar coordinate axes and $\sigma^{(r)}_{RR} = \sigma^{(r)}_1, \sigma^{(r)}_{\Theta\Theta} = \sigma^{(r)}_2$. Equations (165) and (166) then reduce to the single equation

$$\frac{\mathrm{d}\sigma^{(r)}_1}{\mathrm{d}R} + \frac{\sigma^{(r)}_1 - \sigma^{(r)}_2}{R} = 0, \tag{167}$$

which must be coupled with the boundary conditions

$$\sigma^{(r)}_1 = 0 \quad \text{on } R = A, B. \tag{168}$$

We shall use equations (167) and (168) in Section 4.2 in an analysis of the effect of residual stress on the response of a circular cylindrical tube under extension and inflation.

4.2 Extension and inflation of a thick-walled tube

We now return to the problem considered in Section 1.5.2 for an isotropic material and Section 2.1.2 for an orthotropic material, but with residual stresses incorporated. The strain energy may again be written in the form (100). Thus, $\hat{W}(\lambda, \lambda_z, \varphi)$, with λ and λ_z being the azimuthal and axial stretches. We emphasize again that $\hat{W}(\lambda, \lambda_z, \varphi)$ is not in general symmetric in λ and λ_z and that the angle φ may depend on R.

The principal Cauchy stress differences are given (locally) by

$$\sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z}, \quad \sigma_2 - \sigma_1 = \lambda \hat{W}_{\lambda}. \tag{169}$$

Residual stresses associated with the unloaded configuration may be incorporated through $\hat{W}$, in which case the residual stress differences are given by (169) evaluated for $\lambda = \lambda_z = 1$ and subject to $\sigma^{(r)}_3 = 0$, as discussed above. Alternatively, and this is the approach we adopt here, the additional stresses required to deform the material from the unloaded configuration may be accounted for through $\hat{W}$ via (169) with the (in general unknown) residual stresses incorporated separately. We therefore replace (169) by

$$\sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z} + \sigma^{(r)}_3 - \sigma^{(r)}_1, \quad \sigma_2 - \sigma_1 = \lambda \hat{W}_{\lambda} + \sigma^{(r)}_2 - \sigma^{(r)}_1, \tag{170}$$

where $\sigma^{(r)}_1$, $\sigma^{(r)}_2$, $\sigma^{(r)}_3 = 0$ denote the residual principal Cauchy stresses in the unloaded configuration where the terms in $\hat{W}$ vanish. Note that $\sigma^{(r)}_1$ and $\sigma^{(r)}_2$ are independent of the deformation from the unloaded configuration (i.e. they depend only on R).

For the considered cylindrically symmetric deformation the (radial) equilibrium equations for the deformed and unloaded configurations are, respectively,

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0, \qquad \frac{\mathrm{d}\sigma^{(r)}_1}{\mathrm{d}R} + \frac{1}{R}(\sigma^{(r)}_1 - \sigma^{(r)}_2) = 0 \tag{171}$$

in terms of the principal Cauchy stresses. The solution of equation $(171)_1$ should satisfy the boundary conditions

$$\sigma_1 = \begin{cases} -P & \text{on} \quad r = a \\ 0 & \text{on} \quad r = b, \end{cases} \tag{172}$$

corresponding to pressure $P\,(\geq 0)$ on the inside of the tube and zero traction on the outside. The corresponding boundary conditions for the residual stress are (168).

By making use of (58)–(61) together with $(170)_2$ and $(171)_2$, integration of $(171)_1$ and application of the boundary conditions (172) we obtain

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda} \mathrm{d}\lambda + \lambda_z^{-1} \int_A^B \frac{R^2}{r^2} \frac{\mathrm{d}\sigma_1^{(r)}}{\mathrm{d}R} \mathrm{d}R, \tag{173}$$

where, as in (67), the independent variable has been changed from r to λ in the first integral, while in the second integral r^2 is given by $(58)_1$. When there is no residual stress the result (101) is recovered.

Since, from (60), λ_b depends on λ_a, equation (173) provides an expression for P as a function of λ_a when λ_z is fixed provided that the distribution of residual stress is known. In order to hold λ_z fixed an axial load, N say, must be applied to the ends of the tube. This is given by

$$\begin{aligned} N/\pi A^2 = (\lambda_a^2\lambda_z - 1) \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2} \Big(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda \frac{\partial \hat{W}}{\partial \lambda}\Big) \lambda \mathrm{d}\lambda + P\lambda_a^2 \\ + \lambda_z^{-1} \int_A^B (2\sigma_3^{(r)} - \sigma_2^{(r)} - \sigma_1^{(r)}) R\mathrm{d}R/A^2, \end{aligned} \tag{174}$$

and, as for P, this can only be calculated if the residual stress is known. The term in $\sigma_3^{(r)}\ (= 0)$ has been retained so as to maintain the stress difference structure. Note that equation (68) is recovered when there is no residual stress.

The formulas (173) and (174) are valid for a tube with any number of concentric layers and for a general strain energy with the specified symmetry. In general, $\hat{W}$ will be different for each layer, or, at least, the angle φ will be different in each layer. The radial stress is continuous across the boundary between two layers but the circumferential stress is in general discontinuous at such a boundary.

At this point the residual stress distribution is unknown, and, therefore, to proceed further we require some means of estimating it. For this purpose some additional information is needed. One possible approach is to take the opened-up sector of an arterial ring after a radial cut to correspond to the unstressed configuration (Rachev and Hayashi, 1999, Holzapfel et al., 2000) and to investigate the consequences of this assumption. We shall examine some aspects of this in Section 4.4. One alternative, which we shall consider in Section 4.3, is to investigate the consequences of the assumption that the circumferential stress is uniform at the normal physiological pressure.

4.3 Uniform circumferential stress

For simplicity of illustration we restrict attention here to a tube with a single layer, but the analysis (although somewhat more complicated) can easily be carried over to a tube with two or more layers. If the circumferential stress $\sigma_2 = \sigma_{20}$ is assumed to be constant then it follows from the equilibrium equation $(171)_1$ and the boundary conditions (172) that

$$\sigma_{20} = \frac{P_0 a_0}{b_0 - a_0}, \qquad \sigma_{10} = \sigma_{20}\Big(1 - \frac{b_0}{r_0}\Big), \tag{175}$$

where the zero subscript indicates evaluation at the normal physiological pressure (P_0) and

$$r_0^2 = a_0^2 + \lambda_{z0}^{-1}(R^2 - A^2). \tag{176}$$

Use of equations (168), $(170)_2$, $(171)_2$ and (175) then enables the residual radial stress to be calculated explicitly as

$$\sigma_1^{(r)} = \frac{P_0 a_0 b_0}{b_0 - a_0}\frac{1}{2c_0}\log\left(\frac{(r_0 - c_0)(a_0 + c_0)}{(r_0 + c_0)(a_0 - c_0)}\right) - \int_A^R \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi)\frac{\mathrm{d}R}{R}, \tag{177}$$

where $c_0 = (a_0^2 - \lambda_{z0}^{-1}A^2)^{1/2}$. The corresponding residual circumferential stress is then obtained using $(171)_2$. This leads to

$$\sigma_2^{(r)} = \sigma_1^{(r)} - \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi) + \frac{a_0 b_0 P_0}{(b_0 - a_0)r_0}. \tag{178}$$

Once the residual stresses have been calculated for any given form of $\hat{W}$, the pressure P in a general (cylindrically symmetric) configuration can be calculated from (173) and the stresses from (170) and $(171)_1$. Since $\sigma_3^{(r)} = 0$ the axial load N can be obtained from (174).

By applying the boundary condition (168) at $R = B$ to (177) we obtain

$$\frac{P_0 a_0 b_0}{b_0 - a_0}\frac{1}{2c_0}\log\left(\frac{(b_0 - c_0)(a_0 + c_0)}{(b_0 + c_0)(a_0 - c_0)}\right) = \int_A^B \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi)\frac{\mathrm{d}R}{R}. \tag{179}$$

Since, from (176), $b_0^2 = a_0^2 + \lambda_{z0}^{-1}(B^2 - A^2)$, equation (179) provides a connection between the pressure P_0 and the internal radius a_0 (equivalently, $\lambda_{0a} = a_0/A$) for any given value of the axial stretch λ_{z0} and aspect ratio B/A.

A representative plot of the residual stresses is shown in Figure 11 in dimensionless form with the dimensionless stresses defined by

$$\sigma_1^{(r)*} = \sigma_1^{(r)} l/\mu_3, \quad \sigma_2^{(r)*} = \sigma_2^{(r)} l/\mu_3, \tag{180}$$

where $l > 0$ is defined by

$$l = \log\left(\frac{(b_0 - c_0)(a_0 + c_0)}{(b_0 + c_0)(a_0 - c_0)}\right) \tag{181}$$

and μ_3 is the material constant appearing in the strain-energy function (103), which has been used in this calculation with $n = 12$ and $\mu_1^* \equiv \mu_1/\mu_3 = 2$. The axial stretch λ_{z0} has been set at 1.2 and the aspect ratio $B/A = 1.2$. The general qualitative character of the results in Figure 11 is unchanged by using a range of different values of the material parameters.

We observe that the residual radial stress is quite small and is negative except at the boundaries (where it vanishes). The circumferential stress is compressive at the inner boundary and tensile at the outer boundary and is much larger in magnitude than the radial stress. The results shown in Figure 11 are very similar to those reported by, for example, Chuong and Fung (1986) and Takamizawa and Hayashi (1987); see also the discussion in Rachev and Hayashi (1999).

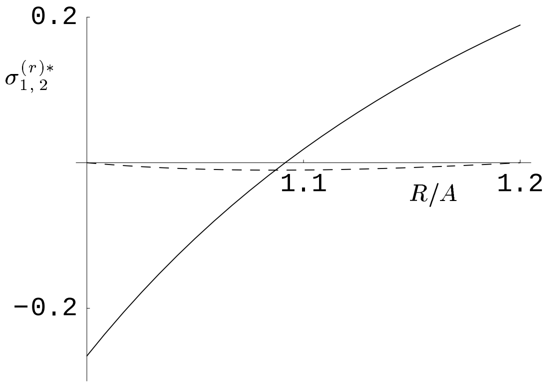

Figure 11. Plot of the dimensionless residual stress distribution for a typical member of the class of anisotropic strain-energy functions (103) based on equations (177) (radial stress – dashed curve) and (178) (circumferential stress – continuous curve).

If, in addition to uniform circumferential stress, it is also assumed that the strain distribution (i.e. λ_0) is constant in the physiological state, then $r_0 = \lambda_0 R$, $\lambda_0^2 \lambda_{z0} = 1$, $c_0 = 0$. Further, if the fibre angle φ is independent of R then equations (177), (178) and (179) simplify to

$$\sigma_1^{(r)}/P_0 = \frac{AB}{B-A}\Big(\frac{1}{A} - \frac{1}{R}\Big) - \frac{\log(R/A)}{\log(B/A)}, \tag{182}$$

$$\sigma_2^{(r)}/P_0 = \frac{B}{B-A} - \frac{1}{\log(B/A)} - \frac{\log(R/A)}{\log(B/A)}, \tag{183}$$

and

$$P_0 = \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi) \log(B/A), \tag{184}$$

respectively (Ogden and Schulze-Bauer, 2000). Plots of (182) and (183) illustrated in the latter paper have the opposite signs to those shown in Figure 11! However, in the work of Rodriguez et al. (1994), which takes account of growth, the residual stress has precisely the structure predicted by (182) and (183) for certain values of their ‘growth stretch’.

4.4 The opening angle method

In Figure 12 an arterial ring in three different configurations is depicted. Figure 12 (b) shows the cross section of an intact artery in the unloaded configuration, while (c) corresponds to an artery subject to internal pressure P. The deformation from (b) to (c) has been discussed in Section 4.3. Here, we focus on the deformation from the opened-up configuration, shown in Figure 12 (a), to the unloaded configuration (b). For reference, we recall that the strain energy associated with the deformation from (b) to (c) is given by $\hat{W}(\lambda, \lambda_z, \varphi)$, where λ_z (constant) is the axial stretch and $\lambda = r/R$ is the circumferential stretch. The fibre angle in (b) is φ.

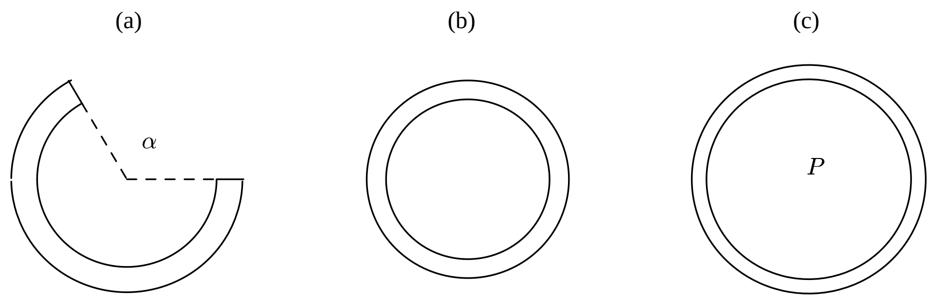

Figure 12. Opened-up configuration of an arterial ring (a), unloaded intact ring (b) and deformed configuration under pressure P (c).

We assume that the sector in (a) is circular and has an opening angle α, as indicated in the figure. Note that different definitions of opening angle are sometimes used in the literature. It is convenient to use the notation

$$k = 2\pi/(2\pi - \alpha), \quad 1 \leq k < \infty, \tag{185}$$

as a measure of the opening angle. In the deformation from (a) to (b) we assume that there is a uniform stretch λ_{zo} induced in the axial direction. The radial part of the deformation is then given by

$$R^2 = A^2 + k^{-1}\lambda_{zo}^{-1}(R_o^2 - A_o^2), \tag{186}$$

where R_o is the radial coordinate in (a) and A_o is the inner radius.

The associated circumferential stretch, denoted λ_o, is

$$\lambda_o = kR/R_o, \tag{187}$$

and we denote by φ_o the fibre angle in (a). (Note that the notations λ_o and λ_{zo} differ from the λ_0 and λ_{z0} used in Section 4.3.) By applying the formula (85) to the deformation (a) to (b) we deduce that

$$\tan\varphi = \lambda_o\lambda_{zo}^{-1}\tan\varphi_o. \tag{188}$$

Next, we assume that the deformation from (a) to (b) is an elastic deformation and described by the strain energy $\hat{W}_o(\lambda_o, \lambda_{zo}, \varphi_o)$, where the subscript o is attached to $\hat{W}$ since, in general, the material response relative to (a) will be different from that relative to (b) even after accounting for the change in fibre angle because, in general, the deformation induces anisotropy in the response relative to (b).

In most analyses it is assumed that the configuration (a) is stress free. We now show that this assumption is valid since *the choice of geometry* necessarily leads to (a) being stress free. Suppose that (a) is not stress free. The geometry ensures that the principal axes of strain are radial and circumferential. Since the deformation is independent of the polar coordinate angle, denoted Θ_o, it follows that the principal axes of stress coincide with those of strain and that the only equilibrium equation not satisfied trivially in (a) is the radial equation

$$\frac{\mathrm{d}\sigma_{o1}^{(r)}}{\mathrm{d}R_o} + \frac{1}{R_o}(\sigma_{o1}^{(r)} - \sigma_{o2}^{(r)}) = 0, \tag{189}$$

where $\sigma_{o1}^{(r)}$ and $\sigma_{o2}^{(r)}$ are, respectively, the radial and circumferential (residual) principal stresses in (a). Since the load must vanish pointwise on the (flat) ends of the opened-up ring we must have $\sigma_{o2}^{(r)} = 0$ on those ends (on which Θ_o is constant). It follows from (189) that $\mathrm{d}(R_o\sigma_{o1}^{(r)})/\mathrm{d}R_o = 0$ on the ends, and hence for all Θ_o. Integration of this and application of the zero traction condition $\sigma_{o1}^{(r)} = 0$ on $R_o = A_o$ shows that $\sigma_{o1}^{(r)} \equiv 0$ and hence, by (189), $\sigma_{o2}^{(r)} \equiv 0$.

This result applies for one layer or for two or more concentric layers, so, in particular, for the case of two layers, the interface must therefore form a perfect geometrical match in the configuration (a). In practice this is unlikely, and indeed, as has been shown in experiments in Professor Holzapfel's laboratory, this is certainly not the case. The length of the outer boundary of the *media* is not in general the same as the length of the inner boundary of the *adventitia* in the opened-up configuration. Moreover, the curvatures of these boundaries are not in general the same. For the media and adventitia to fit together in the opened-up configuration there will necessarily be residual stresses in that configuration. In view of the above analysis such a configuration cannot be described by the geometry discussed above and the deformation from (a) to (b) must depend on Θ_o, and possibly also on the axial coordinate. The analysis associated with this more general geometry is, of course, more complicated than described above and will undoubtedly require numerical treatment. In particular, the plane strain assumption is unlikely to be a good approximation to the real situation for a short length of artery. Work is still in progress on this. Here, our analysis is based on (186).

The residual stress distribution in (b) is governed by equation $(171)_2$, which, on integration, gives

$$\sigma_1^{(r)} = \int_A^R (\sigma_2^{(r)} - \sigma_1^{(r)}) \frac{\mathrm{d}R}{R}, \tag{190}$$

but now the integrand in (190) is given by

$$\sigma_2^{(r)} - \sigma_1^{(r)} = \lambda_o \hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o). \tag{191}$$

Thus, in principle, the residual stress can be calculated. However, this requires some additional information.

First, we note that if B_o denotes the outer radius in (a) then the geometrical quantities in (a) and (b) are related by

$$B^2 = A^2 + k^{-1}\lambda_{zo}^{-1}(B_o^2 - A_o^2). \tag{192}$$

Secondly, by applying the boundary condition $\sigma_1^{(r)} = 0$ on $R = B$ to (190) we obtain

$$\int_A^B \lambda_o \hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o) \frac{\mathrm{d}R}{R} = 0, \tag{193}$$

or, equivalently, by changing the integration variable from R to λ_o using (186) and (187),

$$\int_{\lambda_{ob}}^{\lambda_{oa}} \frac{\hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o)}{\lambda_o^2 \lambda_{zo} - k} \mathrm{d}\lambda_o = 0, \tag{194}$$

where λ_{oa} and λ_{ob} are the values of λ_o on the boundaries $R_o = A_o$ and $R_o = B_o$ respectively.

Since our objective is to calculate the residual stress distribution, we suppose that k, A_o, B_o and λ_{zo} are known. Equations (192) and (194) are then two equations from which to determine

A and B, the latter equation depending on the material properties through $\hat{W}$ and φ_o. Note that A, B, A_o, B_o occur in (194) only through the limits. Once A and B are determined the residual stresses can be calculated from (190) and (191). In this way the opening angle can be related to the residual stress in the unloaded configuration and hence, through the analysis in Section 4.3, the opening angle required to give uniform circumferential stress in the normal physiological state can be determined.

In the above considerations we have not made use of the equation

$$\sigma_3^{(r)} - \sigma_1^{(r)} = \lambda_{zo} \hat{W}_{o\lambda_{zo}}(\lambda_o, \lambda_{zo}, \varphi_o). \tag{195}$$

This is important to note since the zero axial load condition $\sigma_3^{(r)} = 0$ in (b) is not in general compatible with the assumed geometrical transformation from (a) to (b). Thus, (195) must be regarded as giving the stress distribution $\sigma_3^{(r)}$ needed to maintain the cylindrical geometry in (b), in particular uniform λ_{zo}. As is done in some treatments, this problem can be circumvented by setting to zero the total axial load

$$2\pi \int_A^B \sigma_3^{(r)} R \mathrm{d}R \tag{196}$$

so as to determine the value of λ_{zo}. Alternatively, λ_{zo} can be prescribed and $\sigma_3^{(r)}$ calculated from (195) once $\sigma_1^{(r)}$ has been determined by the procedure outlined above.

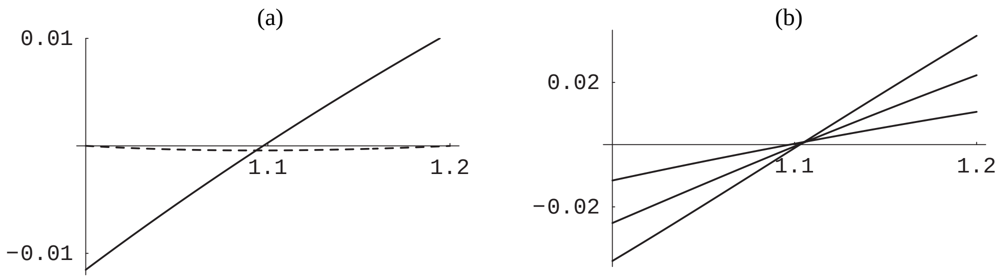

Figure 13. Plot of the residual stresses based on equations (190)–(194) in dimensionless form: (a) radial stress (dashed curve), circumferential stress (continuous curve); $k = 1.5$: (b) comparison of the residual circumferential stresses for $k = 1.5, 1.6, 1.7$.

Some results based on the latter approach are shown in Figure 13 with λ_{zo} set at 1. In Figure 13 (a) dimensionless radial and circumferential residual stresses are plotted against the dimensionless radius R_o/A_o with B_o/A_o set to 1.2. An opening angle of $2\pi/3$, corresponding to $k = 1.5$, has been selected. The plots are based on use of the energy function (103), as for the results shown in Figure 11, and the general character of the results is the same as in Figure 11.

Here, however, the non-dimensionalization used is different and is based on division by the constant $|\mu_3|$. Figure 13 (b) shows a comparison of the circumferential stresses for three different opening angles, corresponding to $k = 1.5, 1.6, 1.7$. Two features should be noted. First, the stress is compressive on the inner boundary and tensile on the outer boundary; second, the maximum magnitudes of the stress increase with the value of k. The point at which the stress vanishes is slightly different for the three curves although this is not apparent on the scale used here. The radial stress likewise increases with k but remains very small compared with the circumferential stress and hence the corresponding comparison is not shown.

If it is not assumed that λ_{zo} is uniform then the problem becomes more difficult because the deformation from (a) to (b) then necessarily involves shearing through the wall thickness. A more general analysis of the opening angle method in which this is accounted for in an axially symmetric setting is given by Ogden (2002).

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